Reports of Enlarged Session of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 22, 2008

VARIATIONAL FORMULATION OF ONE NONLOCAL PROBLEM FOR FORTH ORDER ORDINARY DIFFERENTIAL EQUATION

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Abstract. A nonlocal boundary value problem for a fourth-order ordinary differential equation is considered. Variational formulation of the problem is studied. The minimization of this functional gives a solution of the problem.

Keywords and phrases: Nonlocal problem; ordinary differential equation; variational formulation.

AMS subject classification: 34B05, 35B99.

In [1] Bitsadze and Samarskii studied new mathematical problems with nonlocal boundary conditions. Numerous scientific works (e.g., see [2-16]) deal with the analysis of the problems considered in [1] and some of their generalizations. In these papers, attention was mainly paid to the existence and uniqueness of solutions and to issues related to the approximate solution of nonlocal boundary value problems.

The aim of the present paper is to analyze a nonlocal boundary value problem for an ordinary fourth-order differential equation in the variational setting. Some attempts in this direction were made in [14, 16], where a variational interpretation of some nonlocal boundary value problems for second-order differential equations was given.

In the present note to ensure the positive definiteness of the operator of the nonlocal boundary value problem to be considered. We introduce a special inner product, define a symmetric extension of functions, and construct an appropriate linear manifold.

Let us consider the nonlocal boundary value problem:

$$Au(x) \equiv \left(k_1(x)\, u''(x)\right)'' - \left(k_2(x)\, u'(x)\right)' + k_3(x)\, u(x) = f(x)\,, \quad x \in \left]-a,\ 0\right[, \tag{1}$$

$$u(-a) = 0, \quad u'(-a) = 0, \quad u'(0) = 0, \tag{2}$$

$$\int\limits_{-\xi}^{0} k_2(x)\, u'(x)\, dx - k_1(0)\, u''(0) + k_1(-\xi)\, u''(-\xi) = 0, \tag{3}$$

where $f(x)\,,\ k_3(x) \in C[-a,\ 0]$, $k_1(x) \in C^{(2)}[-a,\ 0]$, $k_2(x) \in C^{(1)}[-a,\ 0]$; $k_1(x) \geq K_1 = Const > 0$, $k_2(x)\,,\ k_3(x) \geq 0$, and $k_3(x) \equiv 0$ for $x \in [-\xi,\ 0]$, here ξ is a given point in the interval $]0,\ a[$.

Note that the boundary condition (3) with $k_2(x) \equiv 0$ and $k_1(x) \equiv Const > 0$ includes the boundary condition of Bitsadze-Samarskii type.

Let $u(x)$ be a solution of problem (1)-(3) in the class $C^{(4)}\,]-a,\ 0[\ \cap\ C^{(2)}\,[-a,\ 0]$. Throughout the paper, we use the ordinary spaces C^k, which consist of k times continuously differentiable functions, as well as the space L_2 of square integrable functions and the Sobolev space W_2^2.

By $D\,[-a,\ 0]$ we denote the linear manifold of all real functions $v(x)$ defined almost everywhere on $[-a,\ 0[$, having a finite boundary value $v\,(0) \in \mathrm{R}$ and satisfying the inclusion $v\,(x) \in L_2\,[-a,\ 0]$.

Note that, to specify a function $v(x) \in D\,[-a,\ 0]$, one should essentially specify the pair $(v\,(x)\,,\ v\,(0))$ $(x \in [-a,\ 0[)$. The functions $v_1\,(x)$ and $v_2\,(x)$ coincide as elements of the linear manifold $D\,[-a,\ 0]$ if $v_1\,(x) = v_2\,(x)$ almost everywhere on $[-a,\ 0[$ $\ v_1\,(0) = v_2\,(0)$.

Throughout the following, for each continuous function $v(x)$ defined on the interval $[-a,\ 0]$, the corresponding element (pair) of the linear manifold $D\,[-a,\ 0]$ is defined to be the pair formed by the restrictions of $v(x)$ to the sets $[-a,\ 0[$ and $\{0\}$, respectively.

On the lineal $D\,[-a,\ 0]$, we define the operator τ of symmetric extension as follows:

$$\tau v\,(x) = \begin{cases} v\,(x) & \text{for} \quad x \in [-a,\ 0]\,, \\ -v\,(-x) + 2v\,(0) & \text{for} \quad x \in\,]0,\ \xi]\,. \end{cases}$$

Note that τ takes each function $v(x) \in D\,[-a,\ 0]$ to a function $\tilde{v}\,(x) = \tau v\,(x)$ defined almost everywhere on $[-a,\ \xi]$ such that the function $\tilde{v}\,(x) - v\,(0)$ is even almost everywhere on $[-\xi,\ \xi]$.

For two arbitrary functions $v\,(x)$, $w\,(x) \in D\,[-a,\ 0]$, we define the inner product

$$[v,\ w] = \int\limits_{-\xi}^{\xi} \int\limits_{-a}^{x} \tilde{v}\,(s)\,\tilde{w}\,(s)\,ds dx. \tag{4}$$

The inner product (4) makes the linear manifold $D[-a,\ 0]$ a pre-Hilbert space. We denote this space by $H\,[-a,\ 0]$ and introduce the norm

$$\|v\|_{_H} = \left(\int\limits_{-\xi}^{\xi} \int\limits_{-a}^{x} \tilde{v}^2\,(s)\,ds dx \right)^{1/2}$$

corresponding to the inner product (4).

In addition, we introduce the norm

$$\|v\| = \left(\|v\|_{L_2}^2 + v^2\,(0) \right)^{1/2}, \tag{5}$$

on $D\,[-a,\ 0]$, where

$$\|v\|_{L_2}^2 = \int\limits_{-a}^{0} v^2\,(x)\,dx.$$

Following statements are true.

Theorem 1. *The norm $\|\cdot\|$ given by (5) is equivalent to the norm $\|\cdot\|_H$.*
Remark 1. It follows from Theorem 1 that $H\,[-a,\ 0]$ is a Hilbert space.
Suppose that the domain of the operator A is the linear manifold $D_A\,[-a,\ 0]$ that consists of the functions $v\,(x) \in H\,[-a,\ 0]$ satisfying the conditions:

$$v\,(x) \in C^{(4)}\,[-a,\ 0]\,,\quad v^{(i)}\,(0) = 0,\quad i = \overline{1,\ 4},\quad v\,(-a) = 0,\quad v'\,(-a) = 0,$$

$$k_1\,(-\xi)\,v''\,(-\xi) + \int\limits_{-\xi}^{0} k_2\,(x)\,v'\,(x)\,dx = 0.$$

Theorem 2. *The linear manifold $D_A\,[-a,\ 0]$ is dense in the space $H\,[-a,\ 0]$.*
Thus, the operator A maps the linear manifold $D_A\,[-a,\ 0]$ dense in the Hilbert space $H\,[-a,\ 0]$ into the same space $H\,[-a,\ 0]$.
Theorem 3. *The operator A is symmetric and positive definite on $D_A\,[-a,\ 0]$.*
We have thereby obtained a standard situation: A is a positive definite operator on the linear manifold $D_A\,[-a,\ 0]$, which is dense in the Hilbert space $H\,[-a,\ 0]$. We follow the well-known scheme [17]. On the lineal $D_A\,[-a,\ 0]$, we introduce the new inner product

$$[v,\ w]_A = [Av,\ w]$$

$$= \int\limits_{-\xi}^{\xi}\int\limits_{-a}^{x} \left[\bar{k}_1(s)\tilde{v}''(s)\tilde{w}''(s) + \bar{k}_2(s)\tilde{v}'(s)\tilde{w}'(s) + \bar{k}_3(s)\tilde{v}(s)\,\tilde{w}(s)\right] dsdx \tag{6}$$

and the corresponding norm

$$\|v\|_A = \left\{\int\limits_{-\xi}^{\xi}\int\limits_{-a}^{x} \left[\bar{k}_1(s)\left(\tilde{v}''(s)\right)^2 + \bar{k}_2(s)\left(\tilde{v}'(s)\right)^2 + \bar{k}_3(s)\tilde{v}^2(s)\right] dsdx\right\}^{1/2}. \tag{7}$$

The inner product (6) makes $D_A\,[-a,\ 0]$ a pre-Hilbert space. Consider the completion $H_A\,[-a,\ 0]$ of this space in the norm (7). One can readily see that the norm (7) is equivalent to the norm of the space $W_2^2\,[-a,\ 0]$. The space $H_A\,[-a,\ 0]$ consists of the elements of $W_2^2\,[-a,\ 0]$ satisfying condition (2).
Let $\alpha \in$ R. Consider the pair $(f\,(x)\,,\ \alpha)$. It determines a unique element $f_\alpha\,(x)$ of the space $H\,[-a,\ 0]$. For each such element, there exists a unique function $u_\alpha\,(x) \in H_A\,[-a,\ 0]$ minimizing the functional

$$F_\alpha\,(v) = [v,\ v]_A - 2\,[f_\alpha,\ v]\,. \tag{8}$$

The function $u_\alpha(x)$ satisfies the identity

$$[u_\alpha,\ v]_A = [f_\alpha,\ v]$$

for arbitrary $v\,(x) \in H_A\,[-a,\ 0]$.
One can readily see that

$$u_\alpha\,(x) = u_0\,(x) + \alpha z\,(x)\,, \tag{9}$$

where $z(x)$ minimizes the functional (8) for the case in which the first component of the pair $(f\,(x)\,,\ \alpha)$ is the zero function on $[-a,\ 0[$, and $\alpha = 1$.

Let us show that we obtain a solution of problem (1)-(3) from the one-parameter family (9) for $\alpha = 0$.

For $\alpha = 0$, the functional (8) has the form

$$F_0\,(v) = 2\xi \int\limits_{-a}^{0} \Big[k_1(x)\ (v''(x))^2 + k_2(x)\ (v'(x))^2 + k_3(x)\, v^2(x) - 2f(x)\, v(x)\Big]\, dx$$

$$+4v\,(0) \int\limits_{-\xi}^{0} \int\limits_{x}^{0} f(s) ds dx. \tag{10}$$

Let us find the variation of the functional (10) on a solution $u(x)$ of problem (1)-(3). After some transformations we have

$$\delta F_0\,(u) = \frac{d}{dt} F_0\,(u+tv)\Big|_{t=0} = 4\xi \int\limits_{-a}^{0} [k_1(x)\, u''(x)\, v''(x) + k_2(x)\, u'(x)\, v'(x)$$

$$+k_3(x)\, u(x)\, v(x)\ -\ f(x)\, v(x)]\, dx + 4v\,(0) \int\limits_{-\xi}^{0} \int\limits_{x}^{0} f(s) ds dx$$

$$= -4\xi\, (k_1 u'')'\,(0)\, v\,(0) + 4\xi v\,(0)\, (k_1 u'')'\,(0)$$

$$+4v\,(0) \left[k_1\,(-\xi)\, u''\,(-\xi) - k_1\,(0)\, u''\,(0) + \int\limits_{-\xi}^{0} k_2\,(x)\, u'\,(x)\, dx\right] = 0.$$

Therefore, the functional (10) attains its minimum on the solution $u(x)$ of problem (1)-(3).

Remark 2. For the minimizing function $u(x)$ of the functional $F_0(v)$, we have

$$\delta F_0(u) = 4\xi \int\limits_{-a}^{0} [k_1(x)u''(x)v''(x) + k_2(x)u'(x)v'(x) + k_3(x)u(x)v(x) - f(x)v(x)]dx$$

$$+4v(0) \int\limits_{-\xi}^{0} \int\limits_{x}^{0} f(s) ds dx = 0, \ \ \forall v(x) \in H_A[-a,\, 0].$$

In particular, the relation

$$\int\limits_{-a}^{0} [k_1(x)u''(x)v''(x) + k_2(x)u'(x)v'(x) + k_3(x)u(x)v(x) - f(x)v(x)]dx = 0$$

holds for an arbitrary infinitely differentiable compactly supported function $v(x)$. It follows that $u(x)$ is a weak solution of equation (1). For an appropriate smoothness of the input data of equation (1), one can also obtain the desired inclusion $u(x) \in C^4] - a,\, 0[\cap C^2[-a,\, 0]$ (e.g., see [17]).

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Received 17.09.2008; accepted 24.12.2008.