Reports of Enlarged Session of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 22, 2008

ON INVESTIGATION AND NUMERICAL SOLUTION OF ONE NONLINEAR BIOLOGICAL MODEL

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Abstract. The two-dimensional system of nonlinear partial differential equations is considered. This system arises in process of vein formation in meristematic tissues of young leaves. Decomposition and variable directions finite difference schemes are studied. Convergence of these schemes are given.

Keywords and phrases: System of nonlinear partial differential equations; decomposition and variable finite difference schemes.

AMS subject classification: 35Q80; 65N06; 65Y99.

Nonlinear systems of partial differential equations describing various processes of diffusion were and are nowadays objects of researching for many scientists. Establishing of qualitative and structural characteristics of initial-boundary value problems of these systems, constructing of discrete analogues and investigation of numerical algorithms are actual and quickly developing parts of modern mathematics.

The main features of many of such systems are expressed in fact that they contain equations of different kinds, which are strongly connected to each other. Mentioned condition for each concrete system determines the use of respective methods of research, because general theory is incompletely developed for such systems even in linear case. Naturally arises the questions of approximate solution of these problems which also are connected with serious complexities as well.

The considered model is connected with process of vein formation in meristematic tissues of young leaves. Mentioned model has the following form [1]:

$$\begin{gathered}\frac{\partial U}{\partial t}=\frac{\partial}{\partial x}\left(V\frac{\partial U}{\partial x}\right)+\frac{\partial}{\partial y}\left(W\frac{\partial U}{\partial y}\right),\\ \frac{\partial V}{\partial t}=-V+f\left(V\frac{\partial U}{\partial x}\right),\quad \frac{\partial W}{\partial t}=-W+g\left(W\frac{\partial U}{\partial y}\right).\end{gathered}\tag{1}$$

Here f and g are given sufficiently smooth functions of their arguments, which satisfy the following conditions: $0<d\le f(r)\le D$, $0<d\le g(s)\le D$, $|f'(r)|<D$, $|g'(s)|<D$, where d and D are constants.

The essential difficulties arise in the process of constructing, investigating and realizing the numerical algorithms for model (1). Besides nonlinearity the complexity of studying such problems are conditioned also by its two-dimensionality. Therefore, naturally arises the question of reduction this problem to easier ones. In particular, it is very important to reduce the two-dimensional problem to the set of one-dimensional problems.

In [1],[2] some qualitative and structural properties of solutions of the boundary-value problems for the system (1) are established. In [2] investigations are carried out for one-dimensional case. The large theoretical and practical importance of the investigation and numerical solution of the initial-boundary value problems for the systems of type (1) is pointed out in [1],[2].

The modern computational techniques give the possibility for the direct numerical solution of the multi-dimensional problems of mathematical physics, but the corresponding algorithms are non-economical and difficult for realization. Therefore arises the question of constructing the economical algorithms for solution of multi-dimensional problems requiring arithmetical operations directly proportional to the number of the space grid points for passage from one time level to next one.

Beginning from the basic works [3],[4] in which the scheme of variable directions were suggested, the methods of constructing of effective algorithms for the numerical solution of the multi-dimensional problems of mathematical physics and the sphere of problems solvable with the help of these algorithms were essentially extended. At present there are some effective algorithms for solving the multi-dimensional problems (see, for example, [5]-[7]). These algorithms mainly belong to the methods of splitting-up or sum-approximation according to their approximate properties. In [8] the new difference schemes belonging to the class of algorithms of variable directions are given.

In the present note one kind of such a scheme for the system (1) is given. We should note that some questions of the convergence of such type scheme as well as average model of sum approximation for the system (1) are discussed in the papers [9]-[11]. The convergence of the difference scheme for one-dimensional analogue of the system is studied in [11].

In the parallelepiped $Q = [0,1] \times [0,1] \times [0,T]$, where T is a given positive constant, consider the system (1) with following boundary and initial conditions:

$$U(x,0,t) = U(x,1,t) = U(0,y,t) = U(1,y,t) = 0, \tag{2}$$

$$U(x,y,0) = U_0(x,y), \quad V(x,y,0) = V_0(x,y), \quad W_0(x,y,0) = W_0(x,y). \tag{3}$$

Let us assume that U_0, V_0 and W_0 are given sufficiently smooth functions, such that $U_0(x,y) \geq d$, $V_0(x,y) \geq d$, $W_0(x,y) \geq d$. Suppose that all necessary consistence conditions are satisfied and there exists the sufficiently smooth solution of the problem (1)-(3). It should be noted that the uniqueness of the solution of the problem (1)-(3) is studied in [10].

Under the conditions on functions f and U_0 from the second equation of the system (1) we have

$$\begin{aligned} V(x,y,t) &= e^{-t}V_0(x,y) + e^{-t}\int\limits_0^t e^{\tau} f\left(V\frac{\partial U}{\partial x}\right)d\tau \\ &\geq e^{-t}d + e^{-t}\int\limits_0^t e^{-\tau} d = c = const > 0. \end{aligned} \tag{4}$$

Analogically we prove the upper boundedness of the function V and similar facts for the function W, i.e.

$$V(x,y,t) < C, \quad c < W(x,y,t) < C, \tag{5}$$

where c and C are positive constants. At last using again restrictions on f, g, V_0, W_0, second equation of the system (1) and estimations (4),(5) it is not difficult to get inequalities:

$$\left|\frac{\partial V}{\partial t}\right| < C, \; \left|\frac{\partial W}{\partial t}\right| < C.$$

Later we shall follow notations from [6]. Introduce on the domain Q the grids:

$$\bar{\omega}_{h\tau} = \bar{\omega}_h \times \bar{\omega}_h \times \omega_\tau,$$

$$\bar{\omega}_{1h\tau} = \bar{\omega}_{1h} \times \omega_\tau,$$

$$\bar{\omega}_{2h\tau} = \bar{\omega}_{2h} \times \omega_\tau,$$

where

$$\bar{\omega}_h = \{(x_i, y_j) = (ih, jh),\; i,j = 0,...,M,\; Mh = 1\},$$

$$\bar{\omega}_{1h} = \left\{(x_i, y_j) = ((i-1/2)h, jh),\; i,j = 1,...,M\;,\; Mh = 1\right\},$$

$$\bar{\omega}_{2h} = \left\{(x_i, y_j) = (ih,\; (j-1/2)h),\; i,j = 1,...,M\;,\; Mh = 1\right\},$$

$$\omega_\tau = \left\{t_k = k\tau,\; k = 0,1,...,K,\; K\tau = T\right\},$$

$$u_x^k = \frac{u_{i+1}^k - u_i^k}{h}, \quad u_{\bar{x}}^k = \frac{u_i^k - u_{i-1}^k}{h}, \quad u_t = \frac{u_i^{k+1} - u_i^k}{\tau}.$$

Let us correspond to problem (1)-(3) following decomposition finite difference scheme:

$$\begin{gathered}\frac{u_1^{k+1} - u_1^k}{\tau} = \sigma_1 (v^{k+1} u_{1\bar{x}}^{k+1})_x + (1-\sigma_1)(v^k u_{1\bar{x}}^k)_x, \\ \frac{v^{k+1} - v^k}{\tau} = -v^{k+1} + f(v^k u_{1\bar{x}}^k),\end{gathered} \tag{6}$$

$$u_1(x_i, y_j, t_k) = u_2\left(x_i, y_j, t_{k+1}\right), \quad u_1(x_i, y_j, 0) = U_0\left(x_i, y_j\right),$$

$$(x_i, y_j) \in \varpi_h, \quad v(x_i, y_j, 0) = V_0(x,y), \quad (x_i, y_j) \in \varpi_{1h},$$

$$u_1(0, y_j, t_{k+1}) = 0, \quad u_1(1, y_j, t_{k+1}) = 0, \quad j = 0,1,...,M, \quad k = 0,1,...,K-1;$$

$$\frac{u_2^{k+1} - u_2^k}{\tau} = \sigma_2 (w^{k+1} u_{2\bar{y}}^{k+1})_y + (1-\sigma_2)(w^k u_{2\bar{y}}^k)_y,$$

$$\frac{w^{k+1}-w^k}{\tau} = -w^{k+1} + g(w^k u^k_{2\bar{y}}), \tag{7}$$

$$u_2(x_i, y_j, t_k) = u_1\left(x_i, y_j, t_{k+1}\right), \quad (x_i, y_j) \in \varpi_h, \quad w(x_i, y_j, 0) = W_0(x, y), \quad (x_i, y_j) \in \varpi_{2h},$$

$$u_2(x_i, 0, t_{k+1}) = 0, \quad u_1(x_i, 1, t_{k+1}) = 0, \quad i = 0, 1, ..., M, \; k = 0, 1, ..., K-1.$$

Here σ_1, $\sigma_2 \in]0; 1]$; functions u_1, u_2 are defined on $\bar{\omega}_{h\tau}$; v, w - on $\varpi_{1h\tau}$ and $\varpi_{2h\tau}$ respectively.

Let us correspond to the problem (1)-(3) following scheme of variable directions:

$$\frac{u_1^{k+1}-u_1^k}{\tau} = (v^{k+1} u^{k+1}_{1\bar{x}})_x + (w^k u^k_{1\bar{y}}) y_y,$$

$$\frac{v^{k+1}-v^k}{\tau} = -v^{k+1} + f(v^k u^k_{1\bar{x}}), \tag{8}$$

$$u_1(x_i, y_j, t_k) = u_2\left(x_i, y_j, t_{k+1}\right), \quad u_1(x_i, y_j, 0) = U_0\left(x_i, y_j\right),$$
$$(x_i, y_j) \in \varpi_h, \quad v(x_i, y_j, 0) = V_0(x, y), \quad (x_i, y_j) \in \varpi_{1h},$$
$$u_1(0, y_j, t_{k+1}) = 0, \quad u_1(1, y_j, t_{k+1}) = 0, \quad j = 0, 1, ..., M, \quad k = 0, 1, ..., K-1;$$

$$\begin{aligned} \frac{u_2^{k+1}-u_2^k}{\tau} &= (v^{k+1} u^{k+1}_{1\bar{x}})_x + (w^{k+1} u^{k+1}_{2\bar{y}})_y, \\ \frac{w^{k+1}-w^k}{\tau} &= -w^{k+1} + g(w^k u^k_{2\bar{y}}), \end{aligned} \tag{9}$$

$$u_2(x_i, y_j, t_k) = u_1\left(x_i, y_j, t_{k+1}\right), \quad (x_i, y_j) \in \varpi_h, \quad w(x_i, y_j, 0) = W_0(x, y), \quad (x_i, y_j) \in \varpi_{2h},$$

$$u_2(x_i, 0, t_{k+1}) = 0, \quad u_1(x_i, 1, t_{k+1}) = 0, \quad i = 0, 1, ..., M, \; k = 0, 1, ..., K-1.$$

Under the sufficiently smoothness of exact solution of the problem (1)-(3) the difference schemes (6), (7) and (8), (9) approximate the problem (1)-(3) with the rate $O(\tau + h^2)$. The following statements take place:

Theorem. The finite difference schemes (6), (7) and (8), (9) converge to the exact solution of the problem (1)-(3) when $\tau \to 0$, $h \to 0$ and for the errors $Z_1 = u_1 - U$, $Z_2 = u_2 - U$, $S_1 = v - V$, $S_2 = w - W$ the following inequality holds

$$\|Z_1\|_{\varpi_h} + \|Z_2\|_{\varpi_h} + \|S_1\|_{\varpi_{1h}} + \|S_2\|_{\varpi_{2h}} \le C(\tau + h^2).$$

Remark. In Theorem C is a positive constant independent on τ and h, norms are discrete analogous of the L_2 norm.

Numerous numerical experiments are done which agree with theoretical conclusions.

REFERENCES

1. Mitchison G.I. A Model for Vein Formation in Higher Plants. Proc. R. Soc. Lond. B., **207**, 1166 (1980), 79-109.

2. Bell J., Cosner C., Bertiger W. Solution for a Flux-Dependent Diffusion Model, SIAM J. Math. Anal., **13**, 5 (1982), 758-769.

3. Peaceman D., Rachford H. The Numerical Solution of Parabolic and Elliptic Differential Equations. J. Soc. Industr. Appl. Math., **3**, 1 (1955), 28-42.

4. Douglas J. On the Numerical Investigation of $u_{xx} + u_{yy} = u_t$ by Implicit Methods. Soc. Industr. Appl. Math., **3**, 1 (1955), 42-65.

5. Janenko N.N. The Method of Fractional Steps for Multi-Dimensional Problems of Mathematical Physics. M.: Nauka, 1967, 169 p. (in Russian).

6. Samarskii A.A. Theory of Difference Schemes. M.: Nauka, 1977, 656 p. (in Russian).

7. Marchuk G.I. The Splitting-up Methods. M.: Nauka, 1988, 264 p. (in Russian).

8. Abrashin V.N. A Variant of the Method of Variable Directions for the Solution of Multi-Dimensional Problems in Mathematical Physics. Differential'nye Uravnenia, **26**, 2 (1990), 314-323 (in Russian).

9. Jangveladze T.A. The Difference Scheme of the Type of Variable Directions for One System of Nonlinear Partial Differential Equations. Proc. I.Vekua Inst. Appl. Math., **42**, (1992), 45-66.

10. Dzhangveladze T.A. (Jangveladze) Avaraged Model of Sum Approximation for a System of Nonlinear Partial Differential Equations. Proc. I.Vekua Inst. Appl. Math., **19**, (1987), 60-73 (in Russian).

11. Jangveladze T.A. Investigation and Numerical Solution of Some Systems of Nonlinear Partial Differential Equations. Rep. Enl. Sess. Sem. I.Vekua Inst. Appl. Math., **6**, 1 (1991), 25-28.

Received 17.09.2008; accepted 24.12.2008.