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ON A CLASS OF SPECIAL FUNCTIONS

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Abstract. The present paper deals with a class of special functions which plays a crucial part in investigation of weighted boundary value problems for the degenerate elliptic Euler-Poisson-Darboux equation.

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1. Introduction. The present paper is devoted to a class of special functions represented in the following integral form

$$M_k(a,b,,j,m) := y^{b+m-k-1} \int\limits_{-\infty}^{+\infty} (\xi - x)^k \frac{\partial^m \theta^j e^{a\theta} \rho^{-b}}{\partial y^m} d\xi, \tag{1}$$

where
$$\theta = \arg(z-\xi), \quad \rho = |z-\xi|^{\frac{1}{2}}, \quad j,k,m \in N^0, \quad z \in R^2_+, \quad \xi \in R^1,$$
a and b are complex constants, R^2_+ is an upper half-plane of the complex plane of the variable $z = x + iy$, R^1 is the axis of the real numbers,

$$\theta \in [0,\pi], \quad N^0 := N \cup \{0\},$$

N is the set of the natural numbers. N_1 and N_2 denote the sets of the odd and even natural numbers, respectively. $N_2^0 := N \cup \{0\}$.

The above class of special functions plays a crucial part in investigation of weighted boundary value problems for the degenerate elliptic equation [1,2]

$$E^{(a,b)}u := y\left(u_{xx} + u_{yy}\right) + au_x + bu_y = 0$$

and iterated one [1,2]
$$\left(\prod_{k=0}^{n-1} E^{(a_k,b)}\right) u = 0,$$
where b, a_k, $k = 0,1,\ldots,n-1$, are, in general, complex constants. When $y = 0$, the above equations have an order degeneration.

2. Main Theorem.

Theorem. *The function $M_k(a,b,,j,m)$ is defined [i.e., the integral (1) exists] and is independent of x, y:*
when

$$Re\, b+m-k-1>0 \tag{2}$$

and either $a\neq 0$, $m\in N^0$, or $a=0$, $j\neq 0$, $m\in N^0$, or $a=j=m=0$, or $a=j=0$, $b\neq 0,-2,\dots,-2\left(m-\left[\frac{m}{2}\right]-1\right)$, $m\in N_2$;
or when

$$Re\, b+m-k>0 \tag{3}$$

and $a=j=0$, $b\neq 0,-2,\dots,-2\left(m-\left[\frac{m}{2}\right]-1\right)$, $m\in N_1$.

If $a=j=0$, and either $b\in\left\{0,-2,\dots,-2\left(m-\left[\frac{m}{2}\right]-1\right)\right\}$, $m\in N$, or condition (3) is fulfilled when $m,k\in N_1$, or (2) is fulfilled when $k\in N_1$, $m\in N_2^0$, then

$$M_k(0,b,0,m)=0. \tag{4}$$

Proof. Using the method of mathematical induction, we prove that

$$\frac{\partial^m e^{a\theta}\rho^{-b}}{\partial y^m}=\sum_{\kappa=1}^{\left[\frac{m}{2}\right]+1} B_\kappa\left(b,m;a(x-\xi),y\right)e^{a\theta}\rho^{-b-2(m-\kappa+1)}, \tag{5}$$

where

$$B_1\left(b,m;a(x-\xi),y\right)=\prod_{i=1}^{m}\left\{a(x-\xi)-\left[b+2(l-1)\right]y\right\}, \tag{6}$$

$$\begin{aligned} &B_\kappa\left(b,m;a(x-\xi),y\right)\\ &\quad=\sum_{\alpha_{\kappa-1}=2\kappa-3}^{m-1}\left(\prod_{j=1}^{\kappa-2}\sum_{\alpha_j=2j-1}^{\alpha_{j+1}-2}\right)\left\{\prod_{k=1}^{\kappa-1}\left[b+2\left(\alpha_k-k\right)\right]\left(\alpha_k-m\right)\right.\\ &\quad\left.\times\prod_{\substack{l=1\\ l\neq\alpha_i-i+1\\ i=1,2,\dots,\kappa-1}}^{m-\kappa+1}\left\{a(x-\xi)-\left[b+2(l-1)\right]y\right\}\right\},\\ &\qquad\kappa=2,\dots,\left[\frac{m}{2}\right]+1, \end{aligned} \tag{7}$$

$$\prod_{j=1}^{\kappa-2}\sum_{\alpha_j=2j-1}^{\alpha_{j+1}-2}:=\sum_{\alpha_{\kappa-2}=2\kappa-5}^{\alpha_{\kappa-1}-2}\sum_{\alpha_{\kappa-3}=2\kappa-7}^{\alpha_{\kappa-2}-2}\cdots\sum_{\alpha_2=3}^{\alpha_3-2}\sum_{\alpha_1=1}^{\alpha_2-2},\ \prod_{j=l}^{l-1}(\cdot)\equiv 1.$$

The last product in (7) we take equal to 1 if none of l are admissible.

It is easy to see that

$$\frac{\partial^m \theta}{\partial y^m} = (x-\xi)^{-m} \left.\frac{\partial^m \theta}{\partial \tau^m}\right|_{\tau=\frac{y}{x-\xi}} = (x-\xi)^{-m} \left.\frac{\partial^m arc\ tg\ \ \tau}{\partial \tau^m}\right|_{\tau=\frac{y}{x-\xi}}$$

$$= (-1)^{m-1}(m-1)!(x-\xi)^{-m}\left[1+\frac{y^2}{(x-\xi)^2}\right]^{-\frac{m}{2}} \sin\left(m\ arc\ tg\frac{x-\xi}{y}\right) \tag{8}$$

$$= (-1)^{m-1}(m-1)!\rho^{-m}\left[sign(x-\xi)\right]^{-m} \sin\left(m\ arc\ tg\frac{x-\xi}{y}\right).$$

Using the method of mathematical induction (with respect to j), we prove that

$$\begin{aligned}\frac{\partial^m \theta^j}{\partial y^m} &= (-1)^{m-j}\left[sign(x-\xi)\right]^{-m}\rho^{-m}\sum_{\kappa_j=0}^{m}\left(\prod_{k=1}^{j-2}\sum_{\kappa_{k+1}=0}^{\kappa_{k+2}}\right)\left\{\binom{m}{\kappa_j}\right. \\ &\times \prod_{k=1}^{j-2}\binom{\kappa_{k+2}}{\kappa_{k+1}}(\kappa_2-1)!\,(m-\kappa_j-1)! \\ &\times \prod_{k=1}^{j-2}(\kappa_{k+2}-\kappa_{k+1}-1)!\sin\left(\kappa_2 arc\ tg\frac{x-\xi}{y}\right) \\ &\times \sin\left[(m-\kappa_j)\,arc\ tg\frac{x-\xi}{y}\right] \\ &\times \prod_{k=1}^{j-2}\sin\left[(\kappa_{k+2}-\kappa_{k+1})\,arc\ tg\frac{x-\xi}{y}\right],\quad j\geq 2.\end{aligned} \tag{9}$$

According to the Leibniz formula,

$$\frac{\partial^m \theta^j e^{a\theta}\rho^{-b}}{\partial y^m} = \sum_{\kappa=0}^{m}\binom{m}{\kappa}\frac{\partial^\kappa \theta^j}{\partial y^\kappa}\frac{\partial^{m-\kappa}e^{a\theta}\rho^{-b}}{\partial y^{m-\kappa}}. \tag{10}$$

By virtue of (5)-(7), (9), it is easy to show that for a fixed z belonging to the closure of any bounded domain from R^2_+, we have:

$$\frac{\partial^m e^{a\theta}\rho^{-b}}{\partial y^m} = O\left(|x-\xi|^{-Re\ b-m}\right),\ |\xi|\to+\infty,\ a\neq 0, \tag{11}$$

$$\frac{\partial^m \rho^{-b}}{\partial y^m} = \begin{cases} O\left(|x-\xi|^{-Re\ b-2\left(m-\left[\frac{m}{2}\right]\right)}\right),\ |\xi|\to+\infty, \\ 0,\ b\in\left\{0,-2,\ldots,-2\left(m-\left[\frac{m}{2}\right]-1\right)\right\},\ m\in N, \end{cases} \tag{12}$$

$$\frac{\partial^m \theta^j}{\partial y^m} = O\left(|x-\xi|^{-m}\right),\ |\xi|\to+\infty,\ j\in N. \tag{13}$$

After substitution $\xi = x+yt$ we get:

$$\left.\frac{\partial^m e^{a\theta}\rho^{-b}}{\partial y^m}\right|_{\xi=x+yt} = O\left(|t|^{-Re\ b-m}\right),\ |t|\to+\infty,\ a\neq 0; \tag{14}$$

$$\left.\frac{\partial^m \rho^{-b}}{\partial y^m}\right|_{\xi=x+yt} = O\left(|t|^{-Re\, b-2\left(m-\left[\frac{m}{2}\right]\right)}\right), \ |t| \to +\infty; \tag{15}$$

$$\left.\frac{\partial^m \theta^j}{\partial y^m}\right|_{\xi=x+yt} = O\left(|t|^{-m}\right), \ |t| \to +\infty, \ j \in N. \tag{16}$$

In view of (11)-(16), in the above - mentioned domain from (10) we obtain

$$\frac{\partial^m \theta^j e^{a\theta} \rho^{-b}}{\partial y^m}$$

$$= \begin{cases} O\left(|x-\xi|^{-Re\, b-m}\right) = O\left(|t|^{-Re\, b-m}\right), \ |\xi|, |t| \to +\infty, \\ \quad \text{when either } a \neq 0, \ m \in N^0, \text{ or } a = 0, \ j \neq 0, \ ; m \in N^0, \\ \quad \text{or } a = j = m = 0, \\ \quad \text{or } a = j = 0, \ b \neq 0, -2, \dots, -2\left(m - \left[\frac{m}{2}\right] - 1\right), \ m \in N_2; \\ O\left(|x-\xi|^{-Re\, b-m-1}\right) = O\left(|t|^{-Re\, b-m-1}\right), \ |\xi|, |t| \to +\infty, \\ \quad \text{when } a = j = 0, \ b \neq 0, -2, \dots, -2\left(m - \left[\frac{m}{2}\right] - 1\right), \ m \in N_1; \\ 0, \ \text{when } a = j = 0, \ b \in \left\{0, -2, \dots, -2\left(m - \left[\frac{m}{2}\right] - 1\right)\right\}, \ m \in N. \end{cases}$$

By virtue of (5)-(7) and (8), (9), we have

$$\begin{aligned} &\left.\frac{\partial^m e^{a\theta} \rho^{-b}}{\partial y^m}\right|_{\xi=x+yt} \\ &= y^{-b-m} \sum_{\kappa=1}^{\left[\frac{m}{2}\right]+1} \tilde{B}_\kappa(b, m; at) e^{a \cdot arc\ ctg(-t)} (1+t^2)^{-\frac{b}{2}-m+\kappa-1}, \end{aligned} \tag{17}$$

where

$$\tilde{B}_1(b, m; at) = (-1)^m \prod_{l=1}^{m} \left[at + b + 2[l-1)\right],$$

$$\begin{aligned} \tilde{B}_\kappa(b, m; at) = (-1)^{m-2\kappa+2} \sum_{\alpha_{\kappa-1}=2\kappa-3}^{m-1} \left(\prod_{j=1}^{\kappa-1} \sum_{\alpha_j=2j-1}^{\alpha_{j+1}-2} \right) \Bigg\{ \prod_{k=1}^{\kappa-1} \left[b + 2\left(\alpha_k - k\right)\right] \left(m - \alpha_k\right) \\ \times \prod_{\substack{l=1 \\ l \neq \alpha_i - i + 1 \\ i=1,2,\dots,\kappa-1}}^{m-\kappa+1} \left[at + b + 2(l-1)\right] \Bigg\}, \qquad \kappa = 2, 3, \dots, \left[\frac{m}{2}\right] + 1; \end{aligned}$$

and

$$\left.\frac{\partial^m \theta^j}{\partial y^m}\right|_{\xi=x+yt}$$

$$= \begin{cases} (-1)^{m-j}\left[sign(-t)\right]^{-m} y^{-m}\left(1+t^2\right)^{-\frac{m}{2}} \sum\limits_{\kappa_j=0}^{m}\left(\prod\limits_{k=1}^{j-2}\sum\limits_{\kappa_{k+1}=0}^{\kappa_k+2}\right)\left\{\begin{pmatrix} m \\ \kappa_j \end{pmatrix}\right. \\ \times \prod\limits_{k=1}^{j-2}\begin{pmatrix} \kappa_{k+2} \\ \kappa_{k+1} \end{pmatrix}(\kappa_2-1)!\,(m-\kappa_j-1)! \\ \times \prod\limits_{k=1}^{j-2}(\kappa_{k+2}-\kappa_{k+1}-1)!\sin\left[\kappa_2 arc\ tg(-t)\right] \\ \times \sin\left[(m-\kappa_j)\,arc\ tg(-t)\right] \\ \left.\times \prod\limits_{k=1}^{j-2}\sin\left[(\kappa_{k+2}-\kappa_{k+1})\,arc\ tg(-t)\right]\right\}, \quad j \geq 2; \\ (-1)^{m-1}(m-1)!y^{-m}\left(1+t^2\right)^{-\frac{m}{2}}\left[sign(-t)\right]^{-m} \\ \times \sin\left[m\ \ arc\ tg(-t)\right], \quad j=1, \end{cases} \tag{18}$$

respectively.

If (3) is fulfilled and $m, k \in N_1$, then

$$M_k(0,b,0,m)$$
$$= y^{b+m-k-1}\int\limits_{-\infty}^{+\infty}(\xi-x)^k\frac{\partial^m\rho^{-b}}{\partial y^m}d\xi = y^{b+m}\int\limits_{-\infty}^{+\infty}t^k\left.\frac{\partial^m\rho^{-b}}{\partial y^m}\right|_{\xi=x+yt}dt = 0,$$

since the integrand because of (17) is an odd function with respect to t while the integral, in view of (3), is convergent. So, (4) is proved.

After substitution $\xi = x + yt$ the expression (1) will get the following form

$$M_k(a,b,j,m) = y^{b+m}\int\limits_{-\infty}^{+\infty}t^k\left.\frac{\partial^m\theta^j e^{a\theta}\rho^{-b}}{\partial y^m}\right|_{\xi=x+yt}dt. \tag{19}$$

Now, according to (10), (17) and (18), it is clear that the right hand side of (19) and, therefore, $M_k(a,b,j,m)$ is independent of x, y.

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