Reports of Enlarged Session of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 22, 2008

ON ONE MIXED BOUNDARY VALUE PROBLEM FOR THE NON-SHALLOW SPHERICAL LAYER

Chokoraia D.

Iv. Javakhishvili Tbilisi State University

Abstract. In the present paper spherical bodies of shells type are discussed, when the displacement vector is independent from the thickness coordinate. The mixed boundary value problem for the spherical cegment has been solved.

Keywords and phrases: Non-shallow shell, displacement vector, stress vector.

AMS subject classification: 74K25.

In the present paper the non-shallow spherical bodies of shell type are discussed, when the displacement vector is independent from the thickness coordinate x_3.

Let suppose that the displacement vector is independent from the thickness coordinate x_3

$$\overrightarrow{u}(x^1, x^2, x^3) = \overrightarrow{u}(x^1, x^2).$$

It is known, that the equilibrium equations and stress-strain relations (Hook's Law) have the following complex form in the system of isometric coordinates [1]

$$\begin{cases} \dfrac{1}{\Lambda}\dfrac{\partial}{\partial z}(T_{11} - T_{22} + 2iT_{12}) + \dfrac{\partial}{\partial \bar{z}}T^{\alpha}_{\alpha} + \dfrac{2}{\rho}T_{+} + F_{+} = 0, \\ \dfrac{1}{\Lambda}\left(\dfrac{\partial T_{+}}{\partial z} + \dfrac{\partial \overline{T}_{+}}{\partial \bar{z}}\right) + \dfrac{1}{\rho}\left(T^{33} - T^{\alpha}_{\alpha}\right) + F_3 = 0. \end{cases} \tag{1}$$

$$\begin{cases} T_{11} - T_{22} + 2iT_{12} = 4\mu\Lambda\dfrac{\partial u^{+}}{\partial \bar{z}}, \\ \theta = \dfrac{1}{\Lambda}\left(\dfrac{\partial u_{+}}{\partial z} + \dfrac{\partial \overline{u}_{+}}{\partial \bar{z}}\right), \quad \Lambda = \dfrac{4\rho^2}{1 + z\bar{z}}, \\ T^{\alpha}_{\alpha} = 2(\lambda + \mu)\left(\theta + \dfrac{2}{\rho}u_3\right), \\ T_{33} = \lambda\left(\theta + \dfrac{2}{\rho}u_3\right) = \dfrac{\lambda}{2(\lambda + \mu)}T^{\alpha}_{\alpha}, \\ T_{+} = T_{13} + iT_{23} = T_{31} + iT_{32} = \mu\left(2\dfrac{\partial u_3}{\partial \bar{z}} - \dfrac{1}{\rho}u_{+}\right), \\ F_{+} = F_1 + iF_2, \quad u_{+} = u_1 + iu_2, \quad u^{+} = u^1 + iu^2, \end{cases} \tag{2}$$

where $x_1 = tg\frac{\theta}{2}\ cos\varphi$, $x_2 = tg\frac{\theta}{2}\ sin\varphi$, are the isometric coordinates on the shell midsurface.

Here we use the notations:

$$\overrightarrow{T}^{\alpha}(x^1,x^2)=\left(1+\frac{x_3}{\rho}\right)^2\overrightarrow{\sigma}^{\alpha}(x^1,x^2,x^3),$$
$$\overrightarrow{T}^{3}(x^1,x^2)=\left(1+\frac{x_3}{\rho}\right)^2\overrightarrow{\sigma}^{3}(x^1,x^2,x^3),$$
$$\overrightarrow{F}(x^1,x^2)=\left(1+\frac{x_3}{\rho}\right)^2\overrightarrow{\phi}(x^1,x^2,x^3),$$

where $\overrightarrow{\sigma}^i$ are contravariant stress vectors, $\overrightarrow{\phi}$ an external force, $\overrightarrow{u}$ the displacement vector, λ and μ are Lame's constants, ρ is a radius of sphere.

The general representations of this system are given in the forms [2]:

$$u_+=2\rho\left[\frac{\partial\chi}{\partial\overline{z}}-\frac{z}{1+z\overline{z}}\overline{\varphi}(z)-\overline{\psi}(z)\right],$$
$$u_3=\chi+\frac{\lambda+3\mu}{4(\lambda+2\mu)}[\varphi(z)+\overline{\varphi}(z)],$$

where $\varphi(z)$ and $\psi(z)$ are holomorphic functions of z and $\chi(z,\overline{z})$ is a solution of the equation $\nabla^2\chi+\frac{2}{\rho^2}\chi=0$, which is expressed with the help of holomorphic functions $f(z)$ by formula [3]

$$\chi(z,\overline{z})=2Re\left[f(z)-\frac{2\overline{z}}{1+z\overline{z}}\int_0^z f(t)dt\right].$$

Let us consider the mixed boundary value problem for the zone of a sphere with radius $\rho_1=\rho sin\vartheta_1,\quad \rho_2=\rho sin\vartheta_2\quad(\rho_1<\rho_2)$. We shall consider the spherical shell the stereographic production of which in the equatorial space gives us the circular ring which is bounded with two concentric circles, with radius $r_1=tg\frac{\theta_1}{2},\ \ r_2=tg\frac{\theta_2}{2}\ \ (r_1<r_2)$ [4].

We have to find the elasticity balance, when some of components of stresses and displacement are marked on the boundary points.

The boundary conditions for the components of the stresses and displacement vector are expressed with the help of holomorphic functions $\varphi(z),\ \ \psi(z),\ \ f(z)$ by formulas

$$\begin{aligned}T_{(ll)}+iT_{(ls)}=&\frac{\mu}{2\rho}\frac{\lambda+\mu}{\lambda+2\mu}[\varphi(z)+\overline{\varphi}(z)]\\&-4\mu\rho\left[\overline{f''}(z)-\frac{z}{1+z\overline{z}}\left(\overline{\varphi'}(z)+\frac{z}{1+z\overline{z}}\overline{\varphi}(z)\right)\right.\\&\left.-\left(\overline{\psi'}(z)+\frac{2z}{1+z\overline{z}}\overline{\psi}(z)\right)\right]\left(\frac{d\overline{z}}{ds}\right)^2,\end{aligned}\tag{3}$$

$$u_3=\chi(z\overline{z})+\frac{\lambda+3\mu}{4(\lambda+2\mu)}[\varphi(z)+\overline{\varphi}(z)].\tag{4}$$

Let us the boundary conditions be equal to constants on the boundary points

$$\begin{cases}T_{(ll)}+T_{(ls)}=P',\quad r=r_1,\\ T_{(ll)}+T_{(ls)}=P'',\quad r=r_2.\end{cases}\tag{5}$$

$$\begin{cases} u_3 = M', & r = r_1, \\ u_3 = M'', & r = r_2. \end{cases} \tag{6}$$

If the functions $\varphi(z)$, $\psi(z)$, $f(z)$ are introduced by series:

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \psi(z) = \sum_{n=0}^{\infty} b_n z^n, \quad f(z) = \sum_{n=0}^{\infty} n c_n z^{n-1},$$

then the systems (3)-(6) we can rewrite as:

$$\begin{aligned} T_{(ll)} + T_{(ls)} = &\frac{\mu}{2\rho} \sum_{n=-\infty}^{\infty} r_1^n \left\{ \frac{\lambda+\mu}{\lambda+2\mu} [a_n e^{in\varphi} + \overline{a}_n e^{-in\varphi}] \right. \\ &+ \frac{(1+r_1^2)^2}{r_1^2} n(n-1)(n-2) \left[\overline{c}_n e^{-i(n-1)\varphi} - \frac{n+(n+1)r_1^2}{(1+r_1^2)^2} \overline{a}_n e^{-in\varphi} \right. \\ &\left.\left. - \frac{n+(n+2)r_1^2}{(1+r_1^2)^2} \overline{b}_n e^{-i(n+1)\varphi} \right] \right\} = P', \quad r = r_1, \end{aligned} \tag{7}$$

$$\begin{aligned} T_{(ll)} + T_{(ls)} = &\frac{\mu}{2\rho} \sum_{n=-\infty}^{\infty} r_2^n \left\{ \frac{\lambda+\mu}{\lambda+2\mu} [a_n e^{in\varphi} + \overline{a}_n e^{-in\varphi}] \right. \\ &+ \frac{(1+r_2^2)^2}{r_2^2} n(n-1)(n-2) \left[\overline{c}_n e^{-i(n-1)\varphi} - \frac{n+(n+1)r_2^2}{(1+r_2^2)^2} \overline{a}_n e^{-in\varphi} \right. \\ &\left.\left. - \frac{n+(n+2)r_2^2}{(1+r_2^2)^2} \overline{b}_n e^{-i(n+1)\varphi} \right] \right\} = P'', \quad r = r_2, \end{aligned} \tag{8}$$

$$\begin{aligned} u_3 = &\sum_{n=-\infty}^{\infty} r_1^n \left\{ \frac{\lambda+3\mu}{4(\lambda+2\mu)} [a_n e^{in\varphi} + \overline{a}_n e^{-in\varphi}] \right. \\ &\left. + \left(\frac{n}{r_1} - \frac{2r_1}{1+r_1^2} \right) (c_n e^{i(n-1)\varphi} + \overline{c}_n e^{-i(n-1)\varphi}) \right\} = M', \quad r = r_1, \end{aligned} \tag{9}$$

$$\begin{aligned} u_3 = &\sum_{n=-\infty}^{\infty} r_2^n \left\{ \frac{\lambda+3\mu}{4(\lambda+2\mu)} [a_n e^{in\varphi} + \overline{a}_n e^{-in\varphi}] \right. \\ &\left. + \left(\frac{n}{r_2} - \frac{2r_2}{1+r_2^2} \right) (c_n e^{i(n-1)\varphi} + \overline{c}_n e^{-i(n-1)\varphi}) \right\} = M'', \quad r = r_2. \end{aligned} \tag{10}$$

When $n = 0$, the solutions of the systems (7)-(10) have the following forms:

$$\begin{aligned} a_0 &= \frac{1}{r_1^4 - r_2^4} \left\{ r_1^2(1-r_2^4) \left(\frac{\lambda+\mu}{\lambda+3\mu} K - \frac{\rho}{\mu} P' \right) + r_2^2(1-r_1^4) \left(\frac{\lambda+\mu}{\lambda+3\mu} K - \frac{\rho}{\mu} P'' \right) \right\}, \\ b_{-1} &= \frac{r_1^4 r_2^4}{r_1^4 - r_2^4} \left\{ \frac{1}{r_1^2} \left(\frac{\lambda+\mu}{\lambda+3\mu} K - \frac{\rho}{\mu} P' \right) - \frac{1}{r_2^2} \left(\frac{\lambda+\mu}{\lambda+3\mu} K - \frac{\rho}{\mu} P'' \right) \right\}, \\ c_1 + \overline{c}_1 &= \frac{(1+r_1^2)(1+r_2^2)}{2(r_2^2 - r_1^2)} (M' - M''). \end{aligned}$$

When $n > 0$, from the systems (7)-(10) we have

$$\begin{aligned} &a_n = 0, \quad n = \pm 1, \pm 2, \dots; \\ &b_n = 0, \quad n = \pm 1, \pm 2, \dots, \quad n \neq -1; \\ &c_n = 0, \quad n = \pm 1, \pm 2, \dots, \quad n \neq 1. \end{aligned}$$

For the components of the stresses and the displacements we obtain:

$$\begin{aligned} T_{(ll)} &= \frac{\mu}{\rho} \frac{1}{r_1^4 - r_2^4} \left\{ \frac{\lambda + \mu}{\lambda + 2\mu} K [r^2 r_1^2 (r_1^2 - r^2) - r^2 r_2^2 (r_2^2 - r^2) + r_1^2 r_2^2 (r_2^2 - r^2)] \right. \\ &\left. + \frac{\rho}{\mu} P' \frac{r_1^2 (r^4 - r_2^4)}{r^2} - \frac{\rho}{\mu} P'' \frac{r_2^2 (r^4 - r_1^4)}{r^2} \right\}, \\ T_{(ls)} &= 0, \end{aligned}$$

$$\begin{aligned} u_3 &= \frac{1}{2(r_1^2 - r_2^2)} \left\{ r_2^2 - r_1^2 - 1 + r_1^2 r_2^2 + \frac{1 - r^2}{1 + r^2} (1 + r_1^2)(1 + r_2^2) M' \right\} \\ &+ \left\{ r_2^2 - r_1^2 + 1 - r_1^2 r_2^2 - \frac{1 - r^2}{1 + r^2} (1 + r_1^2)(1 + r_2^2) M'' \right\}, \end{aligned}$$

where

$$K = M' + M'' - \frac{1 - r_1^2 r_2^2}{r_2^2 - r_1^2} (M' - M'')$$

and

$$T_{(ll)} = \left(1 + \frac{x^3}{\rho}\right) \sigma_{(ll)}, \quad T_{(ls)} = \left(1 + \frac{x^3}{\rho}\right) \sigma_{(ls)}, \quad T_{(ln)} = \left(1 + \frac{x^3}{\rho}\right) \sigma_{(ln)}.$$

Therefore the plane deformation analogous model for the spherical bodies of shell type has been obtained. The mixed boundary value problem for the spherical segment has been solved, when the displacement vector is independent from the thickness coordinate x_3.

R E F E R E N C E S

1. Meunargia T.V. On One Method of Construction of Geometrically and Physically Non-linear Theory of Non-Sallow Shells, Proc. A. Razmadze Math. Inst., **199** (1999), 133-154.

2. Meunargia T.V. On One Method of Calkulation Spherical Shell, Reports of Enlarged Session of the Seminar of I. Vekua Inst. Appl. Math., TSU, **11**, 1 (1996), 3-6.

3. Vekua I.N. Shell Theory: General Methods of Construction, Pitman Advanced Publishing Program, Boston-London-Melbourne, 1985.

4. Muskelishvili N.I. Some Basic Problems of the Mathematical Theory of Elasticity, Noordhoff, Groningen, Holland, 1953.

Received 30.07.2008; revised 18.12.2008; accepted 24.12.2008.