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ON THE STABILIZATION OF SOLUTION AS $t \to \infty$ AND CONVERGENCE OF THE CORRESPONDING FINITE DIFFERENCE SCHEME FOR ONE NONLINEAR INTEGRO-DIFFERENTIAL EQUATION

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Abstract. Large time behavior of solutions and finite difference approximation of a nonlinear integro-differential equations associated with the penetration of a magnetic field into a substance is studied.

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Process of penetration of the magnetic field into a substance is modeled by Maxwell's system of partial differential equations [1]. If the coefficient of thermal heat capacity and electroconductivity of the substance depend on temperature, then Maxwell's system can be rewritten in the integro-differential form [2]. For the one-component magnetic field the one-dimensional case of this model is given by following integro-differential equation:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left[a\left(\int_0^t \left(\frac{\partial U}{\partial x} \right)^2 d\tau \right) \frac{\partial U}{\partial x} \right], \tag{1}$$

where function $a = a(S)$ is defined for $S \in [0, \infty)$.

The existence of solutions of the initial-boundary value problem for the case $a(S) = 1 + S$ and the uniqueness for more general cases are studied in [2]. In [3] the existence and uniqueness properties studied for the case $a(S) = (1 + S)^p$, $0 < p \leq 1$.

In the work [4] some generalization of equations of type (1) is proposed. In particular, assuming the temperature of the considered body to be constant throughout the material, i.e., depending on time, but independent of the space coordinates, the same process of penetration of the magnetic field into the material is modeled by the integro-differential equation, one-dimensional analogue of that has the form [4]:

$$\frac{\partial U}{\partial t} = a\left(\int_0^t \int_0^1 \left(\frac{\partial U}{\partial x} \right)^2 dx d\tau \right) \frac{\partial^2 U}{\partial x^2}. \tag{2}$$

The existence and uniqueness properties of the solutions of the initial-boundary value problems for the equations of (2) type were first studied in the work [5]. Consequently investigation of (1) and (2) type models were continued in a number of other works [6]-[13]. The existence theorems, that are proved in [2],[3],[5], are based on a priori estimates, Galerkin's method and compactness arguments as in [14],[15] for nonlinear parabolic equations.

Note that in [16] and [17] difference schemes for these and such type models were investigated. Difference schemes for a certain nonlinear parabolic integro-differential model similar to (1) and (2) were studied in [18].

The large time behavior to the solutions of the initial-boundary value problems for (1) and (2) type models for the cases $a(S) = (1+S)^p$, $0 < p \leq 1$ and $-1/2 < p < 0$ are studied in [10],[11] respectively. Note that in these works stabilization of the solutions of the problem with nonhomogeneous boundary data on one side of lateral boundary, has the power-like form. It should be noted also that in [12],[13] exponential stabilization of solutions of the problem with homogeneous boundary condition is given.

The purpose of this note is to continue the study of large time behavior of solutions of the first boundary value problem as well as investigation of difference scheme for the equation (2). Here attention is paid to the case $a(S) = 1 + S$.

In the domain $Q = (0,1) \times (0,\infty)$ let us consider the following initial-boundary value problem for the equation (2):

$$U(0,t) = 0, \quad U(1,t) = \psi, \quad t \geq 0, \tag{3}$$

$$U(x,0) = U_0(x), \quad x \in [0,1], \tag{4}$$

where $\psi = Const > 0$ and $U_0 = U_0(x)$ is a given function.

Following statement takes place.

Theorem 1. *If $a(S) = 1 + S$, $U_0 \in H^3(0,1)$, $U_0(0) = 0$, $U_0(1) = \psi$, then for the unique solution of problem (2)-(4) the following estimates hold:*

$$\left|\frac{\partial U(x,t)}{\partial x} - \psi\right| \leq C \exp\left(-\frac{\alpha t}{2}\right), \quad \left|\frac{\partial U(x,t)}{\partial t}\right| \leq C \exp\left(-\frac{\beta t}{2}\right),$$

where $\alpha = Const$, $0 < \alpha < 1$, $0 < \beta = Const < \alpha$.

Note that we use usual Sobolev spaces $H^k(0,1)$. Here and below C denote positive constants independent of t.

As Theorem 1 shows exponential stabilization of solution of the initial-boundary value problem with nonhomogeneous boundary data on one side of lateral boundary takes place as in case of homogeneous conditions on whole boundary.

The following statements are necessary to prove Theorem 1.

Theorem 2. *If $a(S) = 1 + S$, $U_0 \in H^1(0,1)$, $U_0(0) = 0$, $U_0(1) = \psi$, then for the unique solution of problem (2)-(4) the following estimate is true*

$$\|U - \psi x\|_{L_2(0,1)} + \left\|\frac{\partial U}{\partial x} - \psi\right\|_{L_2(0,1)} \leq C \exp\left(-\frac{t}{2}\right).$$

Theorem 3. *If $a(S) = 1 + S$, $U_0 \in H^2(0,1)$, $U_0(0) = 0$, $U_0(1) = \psi$, then for the unique solution of problem (2)-(4) the following estimate is true*

$$\left\|\frac{\partial U(x,t)}{\partial t}\right\|_{L_2(0,1)} \leq C \exp\left(-\frac{\alpha t}{2}\right).$$

In the rectangle $Q_T = (0,1) \times (0,T)$, where T is a positive constant, we consider following problem:

$$\frac{\partial U}{\partial t} - \left[1 + \int\limits_0^t \int\limits_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx d\tau\right] \frac{\partial^2 U}{\partial x^2} = f(x,t), \tag{5}$$

$$U(0,t) = 0, \quad U(1,t) = \psi, \tag{6}$$

$$U(x,0) = U_0(x). \tag{7}$$

where $f = f(x,t)$ and $U_0 = U_0(x)$ are sufficiently smooth given functions of their arguments.

We correspond to the problem (5)-(7) the difference scheme:

$$\begin{gathered} \frac{u_i^{j+1} - u_i^j}{\tau} - \left[1 + \tau h \sum_{l=1}^{M} \sum_{k=1}^{j+1} (u_{\bar{x},l}^k)^2\right] u_{\bar{x}x,i}^{j+1} = f_i^j, \\ i = 1,2,...,M-1; \;\; j = 0,1,...,N-1, \end{gathered} \tag{8}$$

$$u_0^j = 0, \;\; u_M^j = \psi, \;\; j = 0,1,...,N, \tag{9}$$

$$u_i^0 = U_{0,i}, \;\; i = 0,1,...,M. \tag{10}$$

It is not difficult to obtain the following estimation

$$\|u^n\|_h^2 + \sum_{j=1}^{n} \|u_{\bar{x}}^j]|_h^2 \tau \le C, \quad n = 1,2,...,N. \tag{11}$$

The a-priori estimate (11) guarantees the stability of the scheme (8)-(10).

Theorem 4. *If the problem (5)-(7) has a sufficiently smooth unique solution $U = U(x,t)$, then exists the unique solution $u^j = (u_1^j, u_2^j, \ldots, u_{M-1}^j)$, $j = 1,2,\ldots,N$ of the finite difference scheme (8)-(10) and tends to the $U^j = (U_1^j, U_2^j, \ldots, U_{M-1}^j)$ for $j = 1,2,\ldots,N$ as $\tau \to 0$, $h \to 0$ and the following estimate is true*

$$\|u^j - U^j\|_h \le C(\tau + h), \quad j = 1,2,\ldots,N. \tag{12}$$

We now comment on the numerical implementation of the discrete problem (8)-(10). Note that (8) can be rewritten as:

$$\begin{gathered} \frac{1}{\tau} u_i^{j+1} - A\left(\mathbf{u}^{j+1}\right) \frac{u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}}{h^2} - f_i^j - \frac{1}{\tau} u_i^j = 0, \\ i = 1,\ldots,M-1. \end{gathered}$$

where

$$A\left(\mathbf{u}^{j+1}\right) = 1 + \tau h \sum_{\ell=1}^{M} \sum_{k=1}^{j+1} \left(\frac{u_\ell^k - u_{\ell-1}^k}{h}\right)^2.$$

This system can be written in matrix form

$$\mathbf{H}\left(\mathbf{u}^{j+1}\right) \equiv \mathbf{G}\left(\mathbf{u}^{j+1}\right) - \frac{1}{\tau}\mathbf{u}^j - \mathbf{f}^j = 0.$$

The vector $\mathbf{u}$ containing all the unknowns $u_1, \dots, u_{M-1}$ at the level indicated. The vector $\mathbf{G}$ is given by

$$\mathbf{G}\left(\mathbf{u}^{j+1}\right) = \mathbf{T}\left(\mathbf{u}^{j+1}\right)\mathbf{u}^{j+1},$$

where the matrix $\mathbf{T}$ is symmetric and tridiagonal with elements

$$\mathbf{T}_{ir} = \begin{cases} \dfrac{1}{\tau} + 2\dfrac{A}{h^2}, & r = i, \\ -\dfrac{A}{h^2}, & r = i \pm 1. \end{cases}$$

Newton's method for the system is given by

$$\nabla \mathbf{H}\left(\mathbf{u}^{j+1}\right)\Big|^{(n)} \left(\mathbf{u}^{j+1}\Big|^{(n+1)} - \mathbf{u}^{j+1}\Big|^{(n)}\right) = -\mathbf{H}\left(\mathbf{u}^{j+1}\right)\Big|^{(n)}.$$

Theorem 5. *Given the nonlinear system of equations*

$$g_i\left(x_1, \dots, x_{M-1}\right) = 0, \quad i = 1, 2, \dots, M-1.$$

If g_i are three times continuously differentiable in a region containing the solution $\xi_1, \dots, \xi_{M-1}$ and the Jacobian does not vanish in that region, then Newton's method converges at least quadratically (see [19]).

In our case we can write

$$g_i = u_i^{j+1} - \tau A\left(\mathbf{u}^{j+1}\right)\frac{u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}}{h^2} - \tau f_i^j - u_i^j = 0,$$

$$i = 1, \dots, M-1.$$

It is not difficult to check that the Jacobian doesn't vanish. The differentiability is guaranteed, since ∇H is quadratic. Newton's method is costly, because the matrix changes at every step of the iteration. One can use modified Newton (keep the same matrix for several iterations) but the rate of convergence will be slower.

Various numerical experiments are carried out. Received results agree with theoretical researches.

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