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ABOUT STRONGLY AND WEAKLY SEPARABLE STATISTICAL STRUCTURES

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Abstract. In the paper there is constructed Hilbert space of measures. It is proved that, this Hilbert space of measures is the straight sum of Hilbert subspaces. The necessary and enough conditions, are to know when the statistic structures are strongly separable are proved. There are given examples of strongly, weakly and orthogonal statistic structures.

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Let (E, S) measurable space. Here me recall some definitions (see [1]–[5]).

Definition 1. The following object $\{E, S, \mu_a, a \in A\}$ called statistical structure connected with a stochastic system.

Definition 2. A statistical structure $\{E, S, \mu_a, a \in A\}$ connected with a stochastic system is called orthogonal (singular) if

$$(\forall\, i)\ (\forall\, j)\ \big(i \in A \ \&\ j \in A \ \&\ i \neq j \Longrightarrow \mu_i \perp \mu_j\big).$$

Definition 3. A statistical structure $\{E, S, \mu_a, a \in A\}$ connected with a stochastic system is said to be weakly separable, if there exists a family of S-measurable sets $\{X_a, a \in A\}$ such that the relations

$$(\forall\, i)\ (\forall\, j)\ \big(i \in A \ \&\ j \in A\big) \Longrightarrow \mu_i(X_j) = \begin{cases} 0, & \text{if } \ \mathrm{i} \neq \mathrm{j}, \\ 1, & \text{if } \ \mathrm{i} = \mathrm{j} \end{cases}$$

are fulfilled.

Definition 4. A statistical structure $\{E, S, \mu_a, a \in A\}$ connected with a stochastic system is said to be strongly separable, if there exist pairwise disjoint S-measurable sets $\{X_a, a \in A\}$ such that the relation

$$(\forall\, i)\ \big(i \in A \Longrightarrow \mu_i(X_i) = 1 \ \ \forall\, i \in A\big)$$

is fulfilled.

Definition 5. A linear subset of measures $M_H \subset M^\sigma$ is said to be a Hilbert space of measures if

1) one can introduce on M_H a scale product

$$\langle \mu, \nu \rangle, \ \ \mu,\ \nu \in M_H,$$

such that M_H is the Hilbert space and for every mutually singular measures μ and ν, $\mu,\ \nu \in M_H$, the scale product

$$\langle \mu, \nu \rangle = 0;$$

2) if $\nu \in M_H$ and $|f| \leq 1$, then

$$\nu_f(A) = \int\limits_A f(x)\nu(dx) \in M_H,$$

where $f(x)$ is the S-measurable real function, and

$$\langle \nu_f, \nu_f \rangle \leq \langle \nu, \nu \rangle.$$

Note. If statistical structure is strongly separable, then it is weakly separable and orthogonal. Besides if the statistical structure is weakly separable, then it is orthogonal, but not vice-versa.

Example 1. Let $E = R \times R$, there R is the real measurable part or $R \times R$ Borel σ-algebra $\mathcal{B}(R \times R)$ and let me discuss the following multiplications

$$X_a = \begin{cases} -\infty < x < +\infty, \;\; y = a & \text{for } \text{ a} \in (0,1], \\ x = a - 2, \;\; -\infty < y < +\infty & \text{for } \text{ a} \in [2,3]. \end{cases}$$

Then μ_a, $a \in (0,1] \cup [2,3]$ are Gaussian linear of measures on X_a and μ_0 a plane measure on $R \times R$, then the statistic structure

$$\Big\{E = R \times R, S = \mathcal{B}(R \times R), \mu_a, a \in [0,1] \cup [2,3]\Big\}$$

is orthogonal, but it is not weakly separable.

Let H be separable Hilbert space and $\mathcal{B}(H)$ σ-algebra created by H Borel sets. Let mark the scale product as (x,y), $x\,y \in H$, as it is known Gaussian characteristic function on of μ measure has the following form

$$\chi(z) = \int\limits_H e^{(i(z,x)}\mu(dx) = \exp\Big\{i(a,z) - \frac{1}{2}\,(Bz,z)\Big\},$$

where $a \in H$ is the approximate set and B is the correlating operator. Let us discuss Gaussian's measures on $(H, \mathcal{B}(H))$, which are the same with the correlating operators. Let $\{\mu_a, s \in H\}$ be these measures, so we are discussing Gaussian stochastic structure $\{H, \mathcal{B}(H), \mu_a, a \in H\}$.

Feldman–Gaeck proved that Gaussian measures are either inter equivalent or orthogonal. Let us divide $\{\mu_a, a \in H\}$ into the disjunct families, as follows: let the μ_a, placed in the same families, which are equivalent, so $\mu_a \sim \mu_b$ if only

$$\sum_{i=1}^{\infty} \frac{(a-b, e_i)^2}{\lambda_i} < +\infty \;\; \forall\, a \neq b,$$

where $\{e_i\}_{i=1}^{\infty}$ are orthogonaling basises of B correlating operator, and λ_i are the meanings of B operator. In different families there will be placed the μ_a, for which

$$\sum_{i=1}^{\infty} \frac{(a-b, e_i)^2}{\lambda_i} = +\infty \;\; \forall\, a \neq b.$$

So Gaussian measures planed in different families are orthogonal pairs. Let us choose one representative from each family, then we will get Gaussian stochastic orthogonal structure $\{H, \mathcal{B}(H), \mu_a, a \in A\}$.

Let us discuss M_H family of measure of ν measures $\nu(B) = \sum_{a\in A_1} \int\limits_B g_a(x)\mu_a(dx)$, where $g_a(x)$, $\mathcal{B}(H)$ are measurable real functions, $A_1 \subset A$ is the countable subsets of A and

$$\sum_{a\in A_1} \int\limits_H (g_a(x))^2 \mu_a(dx) < +\infty.$$

Let

$$\langle \nu_1, \nu_2 \rangle = \sum_{a\in A_1\cap A_2} \int\limits_H g_a^1(x) g_a^2(x)\mu_a(dx),$$

where

$$\nu_i(B) = \sum_{a\in A_i} \int\limits_B g_a^i(x)\mu_a(dx), \quad i = 1, 2.$$

Theorem 1. *M_H is Hilbert space of measures and M_H is the straight sum of Hilbert spaces $H_2(\mu_a)$, so $M_H = \bigoplus_{a\in A} H_2(\mu_a)$, where $H_2(\mu_a)$ is the family of measures $\nu(A) = \int\limits_A f(x)\mu_a(dx) \quad \forall\, A \in \mathcal{B}(H)$, that $\int\limits_H |f(x)|^2\mu_a(dx) < +\infty$ and $\|\nu\|_{H_2(\mu_a)} = \Big(\int\limits_H |f(x)|^2\mu_a(dx)\Big)^{1/2}$.*

Theorem 2. *Let* card H *is less, than continuum and $M_H = \bigoplus_{a\in A} H_2(\mu_a)$. In order to $\{H, \mathcal{B}(H), \mu_a, a \in H\}$ be orthogonal structure be strongly separable in* (ZFC) & (MA) *theory, it is necessary and enough that the correspondence $f \longleftrightarrow \psi_f$, given by the equality $\int f(x)\nu(dx) = \langle \psi_f, \nu\rangle \ \ \forall\, \nu \in M_H$ be one-to-one ($f \in F$, where f is the set of those f for which $\int f(x)\nu(dx)$ is defined $\forall\, \nu \in M_H$).*

Proof. *Necessity.* As $\{H, \mathcal{B}(H), \mu_a, a \in H\}$ is strongly separable, then there exist such $\mathcal{B}$ measurable X_a sets, that $\mu_a(H - X_a) = 0$ and $\mu_{\mathrm{a}}(\mathrm{X_b}) = 0 \ \ \forall\, \mathrm{a} \neq \mathrm{b}$. Let function $I_{X_a}(x) \in F$ be corresponded with $\mu_a \in H_2(\mu_a)$. Then $\int I_{X_a}(x)\mu_a(dx) = \int I_{X_a}(x) I_{X_a}(x)\mu_a(dx) = \langle \mu_a, \mu_a\rangle$. Let function $f_\psi(x) = f_1(x) I_{X_a}(x)$ be corresponded with $\psi_1 \in H_2(\mu_a)$. Then any $\psi_2 \in H_2(\mu_a)$

$$\int f_{\psi_1}(x)\psi_2(x)(dx) = \int f_{\psi_1}(x) f_2(x) I_{X_a}(x)\mu_a(dx)$$
$$= \int f_\psi(x) f_2(x)\mu_a(dx) = \langle \psi_1, \psi_2\rangle.$$

Let function $f(x) = \sum_{a\in A_f} g_a(x) I_{X_a}(x) \in F$ be corresponded with the measure $\nu \in M_H$ with the following form $\nu = \sum_{a\in A_0} \int g_a(x)\mu_a(dx)$, then $\forall\, \nu_1 \in M_H \quad \nu_1(B) = \sum_{a\in A_1} \int\limits_B g_a^1(x)\mu_a(dx)$, we have

$$\int f(x)\nu_1(dx)$$
$$= \int \sum_{a\in A_1\cap A_2} g_a(x) g_a^1(x)\mu_a(dx) = \sum_{a\in A_1\cap A_2} \int g_a(x) g_a^1(x)\mu_a(dx) = \langle \nu_1, \nu\rangle$$

so the necessity is proved.

Sufficiency. Let $f \in F$ is corresponded with $\nu_f \in M_H$ for which $\int f(x)\nu(dx) = \langle \nu_f, \nu \rangle$, then $\psi_1,\ \psi_2 \in H_2(\mu_a)$ we have

$$\int f_{\psi_1}(x)\psi_2(dx) = \langle \psi_1, \psi_2 \rangle$$

$$= \int f_1(x)f_2(x)\mu_a(dx) = \int f_{\psi_1}(x)f_2(x)\mu_a(dx).$$

So $f_{\psi_1} = f_1$ for almost every μ_a measures and $f_a(x) > 0\ \int f_a^2(x)\mu_a(dx) < +\infty, \quad \mu_a^* = \int f_a(x)\mu_a(dx)$, then $\int f_{\mu_a^*}(x)\mu_b(dx) = \langle \mu_a, \mu_b \rangle = 0\ \ \forall\, b \neq a$. On the other hand $\mu_a(H - X_a) = 0,\ X_a = \{x :\ f_{\mu_a^*}(x) > 0\}$. So the statistic structure $\{H, \mathcal{B}(H), \mu_a, a \in H\}$ is weakly separable. As $\{\mu_a, a \in H\}$ are Borel measures and on $\operatorname{card} H < 2^{\chi_0}$.

Let these probability measures present in ordering form $(\mu_{a_\xi})_{\xi<\omega_a}$, where ω_a is the first ordinal number of H. As the family of measures $\{\mu_{a_\xi}\}_{\xi<\omega_a}$ is weakly separable, then there can be found such S-measured sets $(X_\xi)_{\xi<\omega_a}$, that $(\forall\,\xi)\ (\forall\,\eta)\ \ \xi \in [0, \omega_a[\ \ \&\ \eta \in [0, \omega_a[\Longrightarrow\ \mu_{a_\xi}(X_\eta)\ =\ \begin{cases} 0, & \text{if } \xi \neq \eta \\ 1, & \text{if } \xi = \eta \end{cases}$. Let define the following ω_a order to be the following condition:

1) $(\forall\,\xi)\ (\xi < \omega_a \Longrightarrow B_\xi$ Borel multiplication from $\mathcal{B}(\mathrm{H}))$;

2) $(\forall\,\xi)\ (\xi < \omega_a \Longrightarrow B_\xi \subset X_\xi)$;

3) $(\forall\,\eta_1)\ (\forall\,\eta_2)\ (\eta_1 < \omega_a)\ \&\ (\eta_2 < \omega_a)\ \&\ \eta_1 \neq \eta_2 \Longrightarrow B_{\eta_1} \cap B_{\eta_2} = \varnothing$;

4) $(\forall\,\xi)\ (\xi < \omega_a) \Longrightarrow \mu_{a_\xi}(B_\xi) = 0$

Let $B_0 = X_0$ for $\xi < \omega_a\ (B_\xi)_{\xi<\omega_a}$, then there can be found such Borel subset Y_ξ from H, that $\bigcup\limits_{\eta<\xi} B_\eta \subset Y_\xi$ and $\mu_{\mathrm{a}_\xi}(\mathrm{B}_\xi) = 0$. Let $B_\xi = X_\xi - Y_\xi$ then the sets $(B_\xi)_{\xi<\omega_a}$ are pairwise disjoint sets and the following correlation $\mu_{a_\xi}(B_\xi) = 1$.

So $\{H, \mathcal{B}(H), \mu_a, a \in H\}$ is strongly separable statistic structure. So the sufficiency is proved.

Note. If me identify functions in the F function set, which are coincided in almost every place for $\{\mu_a, a \in H\}$ measures, then between correspondence $f \to \psi_f$ will be one to one.

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