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ON ONE VARIANT OF AXISYMMETRIC NONLINEAR DEFORMATION OF SHELLS OF REVOLUTION UNDER SUPERCRITICAL LOAD

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Abstract. The nonlinear deformations of elastic spherical shells closed at a pole before and after critical loads are considered. A system of differential equations for the solution of the considered problem is obtained. A numerical solution of the system of differential equations is realized in a particular example.

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Deformation of elastic spherical shells closed at a pole under supercritical load is studied in this paper. Consideration of spherical shells closed at a pole brings us to an equations system whose some coefficients become infinity at a pole. Application of approximate analytic methods of solution to such class of problems is not always advisable because of boundedness of domain of their use.

An approach to construction of solution with the help of numerical method is posed in the paper. A stress-strain state of a spherical shell closed at a pole under exterior pressure is investigated both in subcritical and supercritical domains.

Here we will consider deformation of elastic shells of revolution, where deformation components are represented in quadratic variant of nonlinear elasticity theory [1].

A curvilinear coordinate system α, β, γ is related to a shell of revolution which coincide with major curvature lines and exterior normal to median surface. We denote the displacements along these axes by $u_\alpha, u_\beta, u_\gamma$ and the parameters Lame and curvature radii of the median surface of the shell by A, B and R_1, R_2 respectively.

Further we will consider a class of shells for which hypothesis of Kirchhoff-Love are valid, i.e.

$$\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{33} = 0. \qquad (1)$$

According to [2], displacements of shell points can be written in the form:

$$u_\alpha = u(\alpha,\beta) + \gamma\vartheta_1(\alpha,\beta), \quad u_\beta = v(\alpha,\beta) + \gamma\vartheta_2(\alpha,\beta), \quad u_\gamma = w(\alpha,\beta), \qquad (2)$$

where u, v, w are displacements of the points of median surface of the shell; ϑ_1, ϑ_2 are rotation angles of the normal to median surface of the shells in the planes $\beta = const$ and $\alpha = const$ respectively which are determined by equalities

$$\vartheta_1 = -\frac{1}{A}\frac{\partial w}{\partial \alpha} + \frac{u}{R_1}, \quad \vartheta_2 = -\frac{1}{B}\frac{\partial w}{\partial \beta} + \frac{v}{R_2}. \qquad (3)$$

Substituting (2) into the expressions of nonzero deformations $\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}$ [1], expending them into powers of the variable γ and taking only by the first two terms, we can represent them as:

$$\varepsilon_{11} = E_1 + \gamma K_1, \quad \varepsilon_{22} = E_2 + \gamma K_2, \quad \varepsilon_{12} = \Omega + 2\gamma T, \tag{4}$$

where

$$\begin{aligned} E_1 &= e_{11} + \frac{1}{2}(e_{12}^2 + e_{13}^2), & E_2 &= e_{22} + \frac{1}{2}(e_{21}^2 + e_{23}^2), \\ \Omega &= e_{12} + e_{21} + e_{13}e_{23}, & K_1 &= k_{11} + e_{12}k_{12} + e_{13}k_{13}, \\ K_2 &= k_{22} + e_{21}k_{21} + e_{23}k_{23}, & 2T &= k_{12} + k_{21} + k_{13}e_{23} + k_{23}e_{13}. \end{aligned} \tag{5}$$

The values e_{ij}, k_{ij} in expressions (5) are determined by displacements of the median surface u, v, w and rotation angles of the normal ϑ_1, ϑ_2 as follows:

$$\begin{aligned} e_{11} &= \frac{1}{A}\frac{\partial u}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta}v + \frac{w}{R_1}, & e_{22} &= \frac{1}{B}\frac{\partial v}{\partial \beta} + \frac{1}{AB}\frac{\partial B}{\partial \alpha}u + \frac{w}{R_2}, \\ e_{12} &= \frac{1}{A}\frac{\partial v}{\partial \alpha} - \frac{1}{AB}\frac{\partial A}{\partial \beta}u, & e_{21} &= \frac{1}{B}\frac{\partial u}{\partial \beta} - \frac{1}{AB}\frac{\partial B}{\partial \alpha}v, \\ e_{13} &= \frac{1}{A}\frac{\partial w}{\partial \alpha} - \frac{u}{R_1}, & e_{23} &= \frac{1}{B}\frac{\partial w}{\partial \beta} - \frac{v}{R_2}, \\ k_{11} &= \frac{1}{A}\frac{\partial \vartheta_1}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta}\vartheta_2, & k_{22} &= \frac{1}{B}\frac{\partial \vartheta_2}{\partial \beta} + \frac{1}{AB}\frac{\partial B}{\partial \alpha}\vartheta_1, \\ k_{12} &= \frac{1}{A}\frac{\partial \vartheta_2}{\partial \alpha} - \frac{1}{AB}\frac{\partial A}{\partial \beta}\vartheta_1, & k_{21} &= \frac{1}{B}\frac{\partial \vartheta_1}{\partial \beta} - \frac{1}{AB}\frac{\partial B}{\partial \alpha}\vartheta_2, \\ k_{13} &= -\frac{\vartheta_1}{R_1}, & k_{23} &= -\frac{\vartheta_2}{R_2}. \end{aligned} \tag{6}$$

Further we will consider axisymmetric deformation of shells of revolution. In a concrete case for the shells of revolution we have: $\alpha = s,\ \beta = \theta$, where s is the length of the meridian; θ is a central angle in the parallel circle. The Lame coefficients attain the following values $A = 1$, $B = r$, where r is a distance between the considered point of the coordinate plane and rotation axis. At axisymmetric deformation of shells of revolution the values e_{ij}, k_{ij} have the following form:

$$\begin{gathered} e_{11} = \frac{du}{ds} + \frac{w}{R_1}, \quad e_{22} = \frac{\cos\varphi}{r}u + \frac{\sin\varphi}{r}w, \quad e_{13} = \frac{dw}{ds} - \frac{u}{R_1} \\ k_{11} = \frac{d\vartheta_1}{ds}, \quad k_{22} = \frac{\cos\varphi}{r}\vartheta_1, \quad k_{13} = -\frac{\vartheta_1}{R_1}, \\ e_{12} = e_{21} = e_{23} = k_{12} = k_{21} = k_{23} = 0, \end{gathered} \tag{7}$$

where φ is an angle between the normal to coordinate surface and rotation axis.

According [2,3] we get elastic relations and equilibrium equations energetically.

The elastic relations are as follows:

$$\begin{aligned} T_1 &= \frac{Eh}{1-\nu^2}\big[E_1+\nu E_2+\frac{h^2}{12}R_*(K_1+\nu K_2)\big] \\ T_2 &= \frac{Eh}{1-\nu^2}\big[E_2+\nu E_1+\frac{h^2}{12}R_*(K_2+\nu K_2)\big] \\ M_1 &= \frac{Eh^3}{12(1-\nu^2)}\big[K_1+\nu K_2+R_*(E_1+\nu E_2)\big] \\ M_2 &= \frac{Eh^3}{12(1-\nu^2)}\big[K_2+\nu K_1+R_*(E_2+\nu E_1)\big]. \end{aligned} \tag{8}$$

Here h is the thickness of the shell, E is Yung module, ν is Poisson coefficient, $R_* = \frac{1}{R_1}+\frac{1}{R_2}$.

The equilibrium equations have the following form:

$$\begin{gathered} \frac{d(T_1^* r)}{ds} - T_2^*\frac{dr}{ds} + r\left(\frac{N_1^*}{R_1}+q_1^*\right) = 0, \quad \frac{d(N_1^* r)}{ds} - r\left(\frac{T_1^*}{R_1}+\frac{T_2^*}{R_2}-q_3^*\right) = 0, \\ \frac{d(M_1^* r)}{ds} - M_2^*\frac{dr}{ds} - r\big(T_1^* - \frac{M_2^*}{R_2}\big)\vartheta_1 - rN_1^* = 0, \end{gathered} \tag{9}$$

where

$$\begin{gathered} T_1^* = T_1(1+e_{22}) - N_1 e_{13}, \quad T_2^* = T_2(1+e_{11}), \\ N_1^* = N_1(1+e_{22}) - T_1 e_{13}, \quad M_1^* = M_1(1+e_{22}), \quad M_2^* = M_2(1+e_{11}), \\ q_1^* = q_1(1+e_{11}+e_{22}) - q_3 e_{13}, \quad q_3^* = q_3(1+e_{11}+e_{22}) + q_1 e_{13}. \end{gathered} \tag{10}$$

The values T_1, T_2 involved in (8), (10) are normal tangential forces; N_1 is a shearing force; M_1, M_2 are bending moments; q_1, q_3 are the components of the surface load.

Starting from the initial relations we obtain a resolving equations system for axisymmetric deformation of shells of revolution

$$\frac{d\overline{Y}}{ds} = A(s)\overline{Y} + \overline{F}(s,\overline{Y}) + \overline{f}(s), \tag{11}$$

where $\overline{Y} = (T_1, N_1, M_1, u, w, \vartheta_1)^T$, $A(s) = ||a_{ij}||$ is a matrix $(i,j = 1,2,\dots,6)$, $\overline{F} = (F_1, F_2, \dots, F_6)^T$ is a nonlinear vector function, $\overline{f}(s) = (f_1, f_2, \dots, f_6)^T$ is a vector. The elements of the matrix $A(s)$ are determined by mechanical and geometrical characteristics of the shells, and the components of the vector $\overline{f}(s)$ – by projections of the surface load. The values by which the boundary conditions are formulated, are considered as unknowns in equation (11).

As it was mentioned, some coefficients of system (11) turn into infinity at the pole of the shells $s = 0$ and consequently its direct numerical integration is impossible. Therefore similarly to [4], realizing a limit process, we find the limit values of the coefficients of the resolving system at $s \to 0$.

At the pole of spherical shells system (11) gets the form

$$\frac{dT_1}{ds}=0,\quad \frac{dN_1}{ds}=\frac{T_1}{R}+\frac{T_1}{1+e}\left[\frac{D_1}{(1+\nu)D_2D_3}M_1-\frac{2}{(1+\nu)D_3R}T_1\right]$$
$$-\frac{1}{2}q_3\left(1+\frac{e}{1+e}\right),\quad \frac{dM_1}{ds}=0,\quad \frac{du}{ds}=\frac{1}{(1+\nu)D_3}T_1-\frac{2}{(1+\nu)D_3}M_1-\frac{w}{R}, \qquad (12)$$
$$\frac{dw}{ds}=0,\quad \frac{d\vartheta_1}{ds}=\frac{1}{(1+\nu)D_2D_3}M_1-\frac{2}{(1+\nu)D_3R}T_1,$$

where

$$e=\frac{1}{(1+\nu)D_3}\big(T_1-\frac{2}{R}M_1\big),\quad D_1=\frac{Eh}{1-\nu^2},$$
$$D_2=\frac{Eh^3}{12(1-\nu^2)},\quad D_3=D_1-\frac{4D_2}{R^2}.$$

As an example, a problem of deformation of spherical shells subjected to constant exterior pressure q_3 is considered, when the opened contour is rigidly clamped. In this case the boundary conditions at the pole of the shells $s=0$ have the following form

$$N_1=u=\vartheta_1=0,$$

and on the rigidly clamped contour $s=s_N$ they are as follows

$$u=w=\vartheta_1=0.$$

Numerical solution of the given boundary problem is realized with the help of the method offered in [5]. It is based on adding one equation to the initial system of differential equations, formulating a correct boundary problem for this system, using linearization [6] and discrete-orthogonalization [7] methods.

Since the Cauchy problems for linearized equations systems are solved numerically by the Runge-Kuta method, on the first step of calculations we use system (12) at $s=0$ and system (11) at other points. The numerical solution was carried out for the following data: $R=100$; $h=1$; $s_N=40$; $E=10^5$; $\nu=0,3$.

A dependence of the deflection w at the pole of the shell on exterior pressure q_3 is given in the table

w	-0,121	-0,151	-0,175	-0,192	-0,211	-0,228	-0,243	-0,250
- - - - -	- - - - -	- - - - -	- - - - -	- - - - -	- - - - -	- - - - -	- - - - -	- - - - -
q_3	-9,5	-10	-10,5	-11	-11,5	-12	-12,5	-13,15
w	-0,300	-0,500	-0,700	-0,900	-1,100	-1,200	-1,300	-1,400
- - - - -	- - - - -	- - - - -	- - - - -	- - - - -	- - - - -	- - - - -	- - - - -	- - - - -
q_3	-12,62	-11,85	-11,45	-10,26	-10,05	-10,85	-11,63	-12,43

As it is clear from the table, the critical value of the exterior pressure equals $q_3=-13,15$.

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