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AN APPLICATION OF THE BOUNDARY ELEMENT METHOD FOR SOLVING OF THE BOUNDARY VALUE PROBLEMS FOR BINARY MIXTURES

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In this paper a version of linear theory for a body composed of two isotropic materials suggested by Green-Naghdi-Steel is considered. Kelvin's problem is solved in case of plane deformations when in the point of the domain occupied by the binary mixture point force is acting. By integration of solution of this problem the problem for infinite domain when the constant stresses are distributed on the segment is solved. On the basis of the obtained singular solution numerical realizations of different boundary-value problems are carried out for both finite and infinite domains using the boundary element method.

Let $Oxyx_3$ be Cartesian coordinate system. We consider the plane deformation parallel to the plane of mixture of two isotropic elastic materials. The equations of static equilibrium are of the form [1]

$$\begin{cases} \sigma_{xx,x} + \sigma_{yx,y} + \Phi_x = 0, \\ \sigma_{xy,x} + \sigma_{yy,y} + \Phi_y = 0, \end{cases} \tag{1}$$

and the relations corresponding to the Hooke's generalized law will be written as follows

$$\begin{aligned} &\sigma_{xx} = \Lambda\theta + 2Mu_{x,x}, \quad \sigma_{yy} = \Lambda\theta + 2Mu_{y,y}, \\ &\sigma_{x,y} = Au_{y,x} + (B-\Lambda)u_{x,y}, \quad \sigma_{y,x} = Au_{x,y} + (B-\Lambda)u_{y,x}, \end{aligned} \tag{2}$$

where $\sigma_{xx} = (\sigma'_{xx}, \sigma''_{xx})^T$, $\sigma_{yy} = (\sigma'_{yy}, \sigma''_{yy})^T$, $\sigma_{xy} = (\sigma'_{xy}, \sigma''_{xy})^T$, $\sigma_{yx} = (\sigma'_{yx}, \sigma''_{yx})^T$ are column matrices composed of stress tensor components; $u_x = (u'_x, u''_x)^T$, $u_y = (u'_y, u''_y)^T$ are column matrices of the displacement vector, $\varphi_x = (\varphi'_x, \varphi''_x)^T$, $\varphi_y = (\varphi'_y, \varphi''_y)^T$ are components of volume forces; $\theta = u_{x,x} + u_{y,y}$; $\Lambda,\ M\ A,\ B$ are 2×2 symmetric matrices composed of elasticity constants:

$$\Lambda := \begin{pmatrix} \lambda_1 & \lambda_3 \\ \lambda_3 & \lambda_2 \end{pmatrix}, \quad M := \begin{pmatrix} \mu_1 & \mu_3 \\ \mu_3 & \mu_2 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

$$A := M - \lambda_4 S, \quad B = \Lambda + M + \lambda_4 S;$$

where $\dfrac{\partial u_x}{\partial x}$ is denoted by $u_{x,x}$.

Let us consider an infinite plane. The point force $(F_x,\ F_y)$ at the begining of the coordinate system is acting. The problem is called Kelvin's problem and its solution has the form [4]:

$$u_x = (A^*G - xG_{,x})F_x + (-yG_{,x})F_y, \;\; u_y = (-yG_{,y})F_y + (A^*G - yG_{,y})F_y,$$

$$\sigma_{xx} = [(\Lambda + 2M)(A^* - I)G_{,x} - 2MxG_{,xx}]F_x + [\Lambda(A^* - I)G_{,y} - 2MyG_{,xx}]F_y,$$

$$\sigma_{yy} = [\Lambda(A^* - I)G_{,x} - 2MxG_{,yy}]F_x + [(\Lambda + 2M)(A^* - I)G_{,y} - 2MyG_{,yy}]F_y,$$

$$\sigma_{xy} = [A_0G_{,y} - 2MxG_{,xy}]F_x + [B_0G_{,x} - 2MyG_{,xy}]F_y,$$

$$\sigma_{yx} = [B_0G_{,y} - 2MxG_{,xy}]F_x + [A_0G_{,x} - 2MyG_{,xy}]F_y,$$

$$A_0 := (M + \lambda_4 S)A^* - M + \lambda_4 S,$$

$$B_0 := (M - \lambda_4 S)A^* - M - \lambda_4 S,$$

where $G(x,y) := -\dfrac{1}{2\pi}(I + A^*)^{-1}A^{-1}ln(x^2 + y^2)^{\frac{1}{2}}$ is 2×2 matrix.

We also get the expressions for the components of stress tensor.

Integrating the solutions of Kelvin problem we can solve the problem for an infinite domain, when on the interval $|x| \leq a, \; y = 0$ constant stresses $t_x = P_x = (P_x', \; P_x'')^T$ and $t_y = P_y = (P_y', \; P_y'')^T$ are acting.

For displacements we get

$$u_x = (A^*F + yF_{,y})P_x + (-yF_{,x})P_y, \;\; u_x = (-yF_{,x})P_x + (A^*F - yF_{,y})P_y,$$

and for the stresses we obtain:

$$\sigma_{xx} = \{[(\Lambda + 2M)A^* - \lambda]F_{,x} + 2MyF_{,xy}\}\, P_x + \{[\Lambda(A^* - I)F_{,x} + 2MyF_{,yy}\}\, P_y,$$

$$\sigma_{yy} = \{-[2M - \Lambda(A^* - I)]F_{,x} - 2MyF_{,xy}\}\, P_x + \{[(\Lambda + 2M)(A^* - I)F_{,y} - 2MyF_{,yy}\}\, P_y,$$

$$\sigma_{xy} = \{(A_0 + 2M)F_{,y} + 2MyF_{,yy}\}\, P_x + \{B_0F_{,x} - 2M_yF_{,xy}\}\, P_y,$$

$$\sigma_{yx} = \{(B_0 + 2M)F_{,y} + 2MyF_{,xy}\}\, P_x + \{A_0F_{,x} - 2M_yF_{,xy}\}\, P_y,$$

where $F(x,y)$ is 2×2 matrix of the following form

$$F(x,y) = -\frac{1}{2\pi}(I + A^*)^{-1}A^{-1}\left[y\left(arctg\frac{y}{x-a} - arctg\frac{y}{x+a}\right)\right.$$

$$\left. -(x-a)ln\sqrt{(x-a)^2 + y^2} + (x+a)ln\sqrt{(x+a)^2 + y^2}\right].$$

One of the boundary element methods [3] for numerical solution of various boundary-value problems of mixture plane theory in both finite and infinite domains is based on the obtained solutions. In order to solve the considered boundary-value problem the boundary of the given domain is divided into N elements. If the length of the segment is small enough, then we obtain good enough approximation of the contour. It means, that constant normal and tangent stresses (σ_n^i) and (σ_s^i) are acting on the whole length of each element. There are also different two groups of stresses, e.g. for j-th element P_s^i and P_n^i are applied stresses, (σ_n^i) and (σ_s^i) are real stresses. Using obtained solutions and transformation formulas we can calculate real stresses at the middle points of each element

$$\sigma_s^i = \sum_{j=1}^{N} A_{ss}^{ij} P_s^i + \sum_{j=1}^{N} A_{sn}^{ij} P_n^i, \quad \sigma_s^i = \sum_{j=1}^{N} A_{ns}^{ij} P_n^i + \sum_{j=1}^{N} A_{nn}^{ij} P_n^i, \quad i = 1, 2, ..., N,$$

where $A_{ss}^{ij}, ...$ - are influence coefficients of stresses. Thus the problem is reduced to the finding the fiction loads P_s^i and P_n^i taking into account the boundary conditions on the contour. In order to define the influence coefficients preliminary we use the obtained solutions and transformation formulas of Cartesian coordinates.

Using a boundary element method called the method of fiction load various boundary-value problems of mixture plane theory are solved for infinite domain in the case of stress concentration on the hole contour of various shapes. The numerical solutions of these problems are obtained and corresponding graphs are constructed in system MATLAB.

Let's consider the so called problem as an illustration when we have infinite domain with circular hole, the contour is free from load, and at the infinite have one-sided stretch

$$\sigma_{xx}^{\infty} = p = (p^{'}, \, p^{''})^T, \quad p^{'} > 0, \; p^{''} > 0, \quad \sigma_{yy}^{\infty} = \sigma_{xy}^{\infty} = \sigma_{yx}^{\infty} = 0.$$

The analytic solution for polar components of tangential stresses is of the form:

$$\sigma_{\alpha\alpha} = \left\{ I - [I + M(A - \lambda_5 SA^*)^{-1}] cos 2\alpha \right\} p,$$

$$\sigma_{\alpha r} = - \left\{ [I - M(A - \lambda_5 SA^*)^{-1}] sin 2\alpha \right\} p.$$

Taking into account the symmetry of the problem the numerical solutions are found in the quarter of circular boundary, which is divide into 90 element. The numerical solution is in well correspondence with analityc solution. For the different values of the angle α the following quantities $\dfrac{\sigma_{\alpha\alpha}^{'}}{p^{'}}$, $\dfrac{\sigma_{\alpha\alpha}^{''}}{p^{''}}$ and $\dfrac{\sigma_{\alpha r}^{'}}{p^{'}}$, $\dfrac{\sigma_{\alpha r}^{''}}{p^{''}}$ are calculated. The maximal values of the quantities $\dfrac{\sigma_{\alpha\alpha}^{'}}{p^{'}}, \dfrac{\sigma_{\alpha\alpha}^{''}}{p^{''}}$ on $\alpha = \pm\dfrac{\pi}{2}$, and of $\dfrac{\sigma_{\alpha r}^{'}}{p^{'}}$, $\dfrac{\sigma_{\alpha r}^{''}}{p^{''}}$ on $\alpha = \dfrac{\pi}{4}$, $\alpha = \dfrac{3\pi}{4}$, $\alpha = \dfrac{5\pi}{4}$, $\alpha = \dfrac{7\pi}{4}$ are obtained.

The variations of the quantities $\dfrac{\sigma_{\alpha\alpha}^{'}}{p^{'}}$, $\dfrac{\sigma_{\alpha\alpha}^{''}}{p^{''}}$ and $\dfrac{\sigma_{\alpha r}^{'}}{p^{'}}$, $\dfrac{\sigma_{\alpha r}^{''}}{p^{''}}$ on the contour are given on the table 1.

Table 1

α	$\frac{\sigma'_{\alpha\alpha}}{p'}$	$\frac{\sigma''_{\alpha\alpha}}{p''}$	$\frac{\sigma'_{\alpha r}}{p'}$	$\frac{\sigma''_{\alpha r}}{p''}$
6.9813e-002	-1.2626e+000	-8.0721e-001	-2.8710e-002	1.9140e-002
2.7925e-001	-9.7045e-001	-5.7382e-001	-1.3733e-001	9.1553e-002
5.5851e-001	-6.8457e-002	1.4665e-001	-2.4251e-001	1.6167e-001
8.3776e-001	1.1585e+000	1.1267e+000	-2.7399e-001	1.8266e-001
1.1170e+000	2.3372e+000	2.0681e+000	-2.2220e-001	1.4814e-001
1.3963e+000	3.1091e+000	2.6847e+000	-1.0289e-001	6.8595e-002
1.6755e+000	3.2401e+000	2.7893e+000	4.7696e-002	-3.1797e-002
1.9548e+000	2.6906e+000	2.3503e+000	1.8379e-001	-1.2252e-001
2.2340e+000	1.6270e+000	1.5008e+000	2.6402e-001	-1.7601e-001
2.5133e+000	3.7257e-001	4.9891e-001	2.6401e-001	-1.7601e-001
2.7925e+000	-6.9100e-001	-3.5060e-001	1.8378e-001	-1.2252e-001
3.0718e+000	-1.2405e+000	-7.8956e-001	4.7694e-002	-3.1796e-002
3.2114e+000	-1.2626e+000	-8.0721e-001	-2.8710e-002	1.9140e-002
3.3510e+000	-1.1095e+000	-6.8491e-001	-1.0289e-001	6.8593e-002
3.6303e+000	-3.3762e-001	-6.8336e-002	-2.2220e-001	1.4813e-001
3.9095e+000	8.4106e-001	8.7309e-001	-2.7399e-001	1.8266e-001
4.1888e+000	2.0681e+000	1.8531e+000	-2.4251e-001	1.6167e-001
4.4680e+000	2.9701e+000	2.5736e+000	-1.3733e-001	9.1555e-002
4.7473e+000	3.2733e+000	2.8158e+000	9.5858e-003	-6.3905e-003
5.0265e+000	2.8859e+000	2.5064e+000	1.5359e-001	-1.0239e-001
5.3058e+000	1.9253e+000	1.7391e+000	2.5091e-001	-1.6727e-001
5.5851e+000	6.8310e-001	7.4692e-001	2.7198e-001	-1.8132e-001
5.8643e+000	-4.6273e-001	-1.6827e-001	2.1040e-001	-1.4027e-001
6.1436e+000	-1.1638e+000	-7.2827e-001	8.4874e-002	-5.6583e-002
6.2134e+000	-1.2405e+000	-7.8956e-001	4.7694e-002	-3.1796e-002

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