Reports of Enlarged Session of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 21, 2006-2007

# NUMERICAL SOLUTIONS OF SOME BOUNDARY VALUE PROBLEMS OF THEORY OF ELASTICITY BY BOUNDARY ELEMENT METHOD

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Consideration of the influence of the cracks on the wall of the construction to his hardness by quantity and length of the cracks in the underground constructions, for example, in the construction of the tunnels is necessary and of important. Mathematical model of this practical problem is a boundary value problem, which is considered for infinite body containing single or some cracks originating at the boundary of the internal elliptical hole. The body is homogeneous, isotropic and of plane deformed state. Its inner surface is stress less and all-around tension is given at infinity.

In the present article a corresponding plane boundary value problem (two-dimensional problem) is considered for the domain containing single crack, because the boundary element method [1], used by us may be generalized to solve the problem in the domain containing several cracks.

Analogical practical problem is also solved for elliptical ring with cut, when conditions of symmetry on the sides of the cut are realized [2].

**I.** Find a solution of a system of equilibrium equations[3]:

$$\begin{aligned} &\frac{\partial D}{\partial \theta} - \frac{\partial K}{\partial \alpha} = 0, \quad \frac{\partial \overline{u}}{\partial \theta} + \frac{\partial \overline{v}}{\partial \alpha} = \frac{æ - 1}{æ\mu} h_0^2 D, \\ &\frac{\partial D}{\partial \alpha} + \frac{\partial K}{\partial \theta} = 0, \quad \frac{\partial \overline{v}}{\partial \theta} - \frac{\partial \overline{u}}{\partial \alpha} = \frac{1}{\mu} h_0^2 K, \end{aligned} \tag{1}$$

at infinity domain $\Omega = \{\theta_1 < \theta < \infty,\ 0 < \alpha < 2\pi\}$, the boundary of which $\theta = \theta_1$ contains one radial crack with length $L$ satisfying the following boundary conditions:

$$\theta = \theta_1: \quad \sigma_{\theta\theta} = 0,\ \sigma_{\theta\alpha} = 0, \tag{2}$$

$$\theta \to \infty: \quad \sigma_{\theta\theta} = p,\ \sigma_{\theta\alpha} = 0, \tag{3}$$

$$\alpha = 0, 2\pi: \quad \sigma_{\alpha\alpha} = 0,\ \sigma_{\theta\alpha} = 0,\ (\theta_1 < \theta < \theta_1 + L), \tag{4}$$

where $\overline{u} = \dfrac{2hu}{c^2}$, $\overline{v} = \dfrac{2hv}{c^2}$, $h_0 = \sqrt{\cosh(2\theta) - \cos(2\alpha)}$, $æ = 4(1-\nu)$, $\mu = \dfrac{E}{2(1-\nu)}$; $\nu$, $E$ are well known constants, u, v are components of the displacement vector, and $\sigma_{\theta\theta}$, $\sigma_{\alpha\alpha}$, $\sigma_{\theta\alpha}$ are those of the stress tensor in system of elliptical coordinates $\theta$, $\alpha$ $(0 \le \theta < \infty,\ 0 \le \alpha < 2\pi)$. If $x$, $y$ are Cartesian coordinates, then $x = c\cosh\theta\cos\alpha$, $y = c\sinh\theta\sin\alpha$. $h_\theta = h_\alpha = h = \dfrac{c}{\sqrt{2}}\sqrt{\cosh(2\theta) - \cos(2\alpha)}$ is Lamè's coefficients, $c = 1$ is scale coefficient. To obtain numerical solution of this problem the combined method of fiction load and displacement discontinuity methods [1] is used.

This boundary value problem is generalize of boundary value problem for infinity body containing circular hole and radial cracks [4].

If the boundary of domain is divided by segments $N$ of small lengths, then it is possible to mean, that on every element $i$ constant normal and tangential stresses $\sigma^i_{\theta\theta} = -p$ (or $\sigma^i_{\alpha\alpha} = -p$) and $\sigma^i_{\theta\alpha} = 0$, accordingly, act. Thus boundary conditions (2), (3), (4) in additional stresses will be written in the following form:

$$\theta = \theta_1 : \quad \sigma^i_{\theta\theta} = -p, \quad \sigma^i_{\theta\alpha} = 0 \tag{5}$$

$$\alpha = 0, 2\pi : \quad \sigma^i_{\alpha\alpha} = -p, \quad \sigma^i_{\theta\alpha} = 0, \quad (\theta_1 < \theta < \theta_1 + L). \tag{6}$$

Let continuously distributed point forces correspond to each boundary element, for example, continuously distributed constant stresses $P^j_s$ and $P^j_n$ will correspond to element $j$. Besides for element $j$ there exist applied fictitious stresses $P^j_s$ and $P^j_n$, and real stresses $\sigma^j_s$, $\sigma^j_n$ which arose because of acting stresses applied to every boundary element.

Using solution of Kelvin problem [5] and transformation formulae [6] we can count real stresses $\sigma^i_s \equiv \sigma^i_{\theta\alpha}$ and $\sigma^i_n \equiv \sigma^i_{\theta\theta}$ $(i = 1, \ldots, N_1)$ in the middle point of every element. Thus we obtain the following relations:

$$\sigma^i_s \equiv \sigma^i_{\theta\alpha} = \sum_{j=1}^{N_1} \left( A^{ij}_{ss} P^j_s + A^{ij}_{sn} P^j_n \right), \quad \sigma^i_n \equiv \sigma^i_{\theta\theta} = \sum_{j=1}^{N_1} \left( A^{ij}_{ns} P^j_s + A^{ij}_{nn} P^j_n \right), \tag{7}$$

where $A^{ij}_{ss}$, $A^{ij}_{sn}$, $A^{ij}_{ns}$, $A^{ij}_{nn}$ are boundary coefficients of influence of stresses.

To solve the problem of the cracks the interrupt displacement discontinuity method is created [1]. This method it is based on analytic solution, concerning to displacement discontinuity of constant value on the finite segment of the infinite plane. The analytical solution of this problem is obtained by Crouch [7].

If the crack is divided by segments $N_2 = N - N_1$ of small lengths, then it may be considered that in limits of each segment (element) the displacement discontinuity is constant. Influence of elementary displacement discontinuity of the separate on the displacement and stress of any point of infinite solid body is determined from the analytical solutions obtained by Crouch [7]. For example, in the midpoint of the element i the tangential and the normal stresses may be expressed by the components of the displacement discontinuity of element $j$. Thus we obtain the following relations:

$$\sigma^i_s \equiv \sigma^i_{\theta\alpha} = \sum_{j=N_1+1}^{N} \left( C^{ij}_{ss} D^j_s + C^{ij}_{sn} D^j_n \right), \quad \sigma^i_n \equiv \sigma^i_{\alpha\alpha} = \sum_{j=N_1+1}^{N} \left( C^{ij}_{ns} D^j_s + C^{ij}_{nn} D^j_n \right), \tag{8}$$

where $C^{ij}_{ss}$, $C^{ij}_{sn}$, $C^{ij}_{ns}$, $C_{nn}$ are boundary coefficients of stress influences, $i = N_1 + 1, \ldots, N$.

For satisfaction of the boundary conditions for boundary $\theta = \theta_1$ and for cracks formulas (7) and (8) are used respectively. Thus, we obtain the following system of $2N$

$(N = N_1 + N_2)$ linear algebraic equations:

$$\begin{cases} \sum\limits_{j=1}^{N_1} \left(A_{ss}^{ij} P_s^j + A_{sn}^{ij} P_n^j\right) + \sum\limits_{j=N_1+1}^{N} \left(C_{ss}^{ij} D_s^j + C_{sn}^{ij} D_n^j\right) = 0, \\ \sum\limits_{j=1}^{N_1} \left(A_{ns}^{ij} P_s^j + A_{nn}^{ij} P_n^j\right) + \sum\limits_{j=N_1+1}^{N} \left(C_{ns}^{ij} D_s^j + C_{nn}^{ij} D_n^j\right) = -p, \quad i = 1, \dots, N. \end{cases} \tag{9}$$

In (9) $P_s^j$ and $P_n^j$ are unknown fictitious quantities, and $D_s^j$ and $D_n^j$ are unknown displacement discontinuities.

Received numerical solution of boundary value problem (1), (2), (3), (4)(or (1), (5),(6)).

**II.** Find solution of the system of equilibrium equations (1) in the domain $\Omega_1 = \{\theta_1 < \theta < \theta_2,\ 0 < \alpha < \pi\}$, which satisfies the following boundary conditions:

$$\begin{array}{ll} \text{a) } \theta = \theta_1 : \ \sigma_{\theta\theta} = \mathrm{p}\cos\frac{\alpha}{2},\ \sigma_{\theta\alpha} = 0, & \text{b) } \theta = \theta_2 : \ \sigma_{\theta\theta} = 0,\ \sigma_{\theta\alpha} = 0, \\ \text{c) } \alpha = 0 : \ \mathrm{v} = 0,\ \sigma_{\theta\alpha} = 0, & \text{d) } \alpha = \pi : \ \mathrm{u} = 0,\ \sigma_{\alpha\alpha} = 0. \end{array} \tag{10}$$

Such a boundary value problem is solved by the fiction load method. The components of stresses are calculated in some typical points of the considered domain.

**III.** By the method of separation of variables analytical (exact) solution of problem (1), (10) (when $\theta_2 \to \infty$) is obtained. It is expresed by means of harmonic functions $\varphi_1$, $\varphi_2$. The components of stress tensor have the following form:

$$\begin{aligned} \frac{h^2}{\mu}\sigma_{\theta\theta} = &\left(2\sinh^2\theta_1 \coth\theta \frac{\partial^3\varphi_1}{\partial\alpha^3} - \frac{\partial^2\varphi_1}{\partial\theta\partial\alpha} - 2\frac{\partial^2\varphi_2}{\partial\alpha^2}\right)\sinh\theta\cos\alpha - \\ &- \left(2\cosh^2\theta_1 \tanh\theta \frac{\partial^3\varphi_1}{\partial\theta\partial\alpha^2} - \frac{\partial^2\varphi_1}{\partial\alpha^2} - 2\frac{\partial^2\varphi_2}{\partial\theta\partial\alpha}\right)\cosh\theta\sin\alpha - \\ &- \frac{2\left[\cosh(2\theta_1) - \cosh(1\theta)\right]}{\cosh(2\theta) - \cos(2\alpha)}\left(\frac{\partial^2\varphi_1}{\partial\alpha^2}\cosh\theta\sin\alpha - \frac{\partial^2\varphi_1}{\partial\theta\partial\alpha}\sinh\theta\cos\alpha\right), \end{aligned}$$

$$\begin{aligned} \frac{h^2}{\mu}\sigma_{\alpha\alpha} = &\left(2\cosh^2\theta_1 \tanh\theta \frac{\partial^3\varphi_1}{\partial\theta\partial\alpha^2} + 3\frac{\partial^2\varphi_1}{\partial\alpha^2} - 2\frac{\partial^2\varphi_2}{\partial\theta\partial\alpha}\right)\cosh\theta\sin\alpha - \\ &- \left(2\sinh^2\theta_1 \coth\theta \frac{\partial^3\varphi_1}{\partial\alpha^3} + 3\frac{\partial^2\varphi_1}{\partial\theta\partial\alpha} - 2\frac{\partial^2\varphi_2}{\partial\alpha^2}\right)\sinh\theta\cos\alpha + \\ &+ \frac{2\left[\cosh(2\theta_1) - \cosh(1\theta)\right]}{\cosh(2\theta) - \cos(2\alpha)}\left(\frac{\partial^2\varphi_1}{\partial\alpha^2}\cosh\theta\sin\alpha - \frac{\partial^2\varphi_1}{\partial\theta\partial\alpha}\sinh\theta\cos\alpha\right), \end{aligned}$$

$$\frac{h^2}{\mu}\sigma_{\theta\alpha} = -\left(2\cosh^2\theta_1 \tanh\theta \frac{\partial^3\varphi_1}{\partial\alpha^3} - \frac{\partial^2\varphi_1}{\partial\theta\partial\alpha} - 2\frac{\partial^2\varphi_2}{\partial\alpha^2}\right)\cosh\theta\sin\alpha -$$

$$
-\left(2\sinh^2\theta_1\coth\theta\frac{\partial^3\varphi_1}{\partial\theta\partial\alpha^2}-\frac{\partial^2\varphi_1}{\partial\alpha^2}-2\frac{\partial^2\varphi_2}{\partial\theta\partial\alpha}\right)\cosh\theta\sin\alpha+
$$

$$
+\frac{2\left[\cosh(2\theta_1)-\cosh(1\theta)\right]}{\cosh(2\theta)-\cos(2\alpha)}\left(\frac{\partial^2\varphi_1}{\partial\alpha^2}\sinh\theta\cos\alpha+\frac{\partial^2\varphi_1}{\partial\theta\partial\alpha}\cosh\theta\sin\alpha\right),
$$

where if $\varphi_1=\sum\limits_{j=1}^{2}A_{1j}e^{-k(\theta-\theta_2)}\sin(k\alpha),\ \varphi_2=\sum\limits_{j=1}^{2}A_{2j}e^{-k(\theta-\theta_2)}\cos(k\alpha),\ k=\dfrac{2j-1}{2}$, then the boundary conditions (10c,d) are satisfied.

The unknown $A_{ij}$ $(i,j=1,2)$ values are determined as a result of satisfaction of the boundary conditions (10a,b).

If should be noted, that continuously continuation of the solutions obtained in domain $\Omega_1$ is possible toward the boundary $\alpha=\pi$, that will result in obtaining the domain $\Omega_2=\{\theta_1<\theta<\theta_2,\ 0<\alpha<2\pi\}$ containing cut. On the sides of a cut $\alpha=0$ and $\alpha=2\pi$ conditions of symmetry $v=0$, $\sigma_{\theta\alpha}=0$ are satisfied [2].

The solution obtained by the boundary element method (II item)for greater meaning of $\theta_2$ and the analytical (exact) solution (III item) are coincided with high exactness as well as for circular ring [4]. It is indicated to the fact that using the boundary element method to solve the above presented problems is advisability and is justify and the obtained result is reliable.

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Received 7.06.2006; revised 22.02.2007; accepted 25.12.2007.