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ON THE APPLICATION OF RITZ'S EXTENDED METHOD FOR SOME ILL-POSED PROBLEMS

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Let H be a Hilbert space and $K : H \to H$ be compact, selfadjoint and one-to-one operator, having everywhere dense range. Let us assumed that

$$K(u) = \sum_{k=1}^{\infty} \sigma_k (u, \varphi_k)\varphi_k, \ \sigma_k \to 0, \ \sigma_k > 0 \ ,$$

where $\{\varphi_k\}$ is an orthonormal sequence of eigenfunctions of K with the eigenvalues σ_k, $(\cdot, \cdot)$ denotes the inner product in H. It is easy to prove that under above conditions the system $\{\varphi_k\}$ is complete. The operator K is only positive, while for any $u \in H$ we have that $(Ku, u) \geq 0$. As an operator K it is possible to consider the inverse to an operator A having a discrete spectrum [3]. It is well known that K^{-1} is selfadjoint operator with the purely pointwise spectrum [4] and with the everywhere dense domain of definition. We have $K^{-1}u = \sum_{k=1}^{\infty} \sigma_k^{-1}(u, \varphi_k)\varphi_k$, where the sequence $\{\sigma_k^{-1}\}$ is unbounded and convergent to infinity. It follows that the operator K^{-1} has a discrete spectrum ([3], p. 220). For the operator K^{-1} it is possible apply the received in [2] results. In particular, we can define the operator $K^{-\infty} = A^{\infty}$. To this end let us consider the Frechet space $D(A^{\infty}) = D(K^{-\infty}) = \bigcap_{n=1}^{\infty} D(K^{-n+1})$, the Hilbert norms

$$\|f\|_n^2 = \|f\|^2 + \|K^{-1}f\|^2 + \cdots + \|K^{-n+1}f\|^2 \tag{1}$$

of which are given by the sequence of inner products

$$(x, y)_n = (x, y) + (K^{-1}x, K^{-1}y) + \cdots + (K^{-n+1}x, K^{-n+1}y), \ x, y \in D(K^{-\infty}). \tag{2}$$

It is well known that any Frechet space is isomorphic to a subspace of some product of Banach spaces. In this case the Frechet space $D(K^{-\infty})$ is isomorphic to the subspace M of the Frechet space H^N and this isomorphism is realized by the correspondence

$$x \in D(K^{-\infty}) \to Orb(K^{-1}, x) = \{x, K^{-1}x, \cdots, K^{-n+1}x, \cdots\} \in M \subset H^N. \tag{3}$$

The topology of the space H^N is given by the seminorms

$$\|f\|_n^2 = \|f_1\|^2 + \cdots + \|f_n\|^2, \ f = \{f_k\} \in H^N.$$

Let us define the operator $K^{-\infty}x = \{K^{-1}x, \cdots, K^{-n}x, \cdots\}$. This operator is continuous since it is defined on the whole Frechet space $D(K^{-\infty})$. It is symmetric and positive definite on the space $D(K^{-\infty})$ since for arbitrary $n \in N$ and $x, y \in D(K^{-\infty})$ the following relations hold

$$(K^{-\infty}x, y)_n = (x, K^{-\infty}y)_n \ , \qquad (K^{-\infty}x, x)_n \geq C_n(x, x)_n \ , \tag{4}$$

where

$$[x]_n =: (K^{-\infty}x, x)_n = (K^{-1}x, x) + (K^{-2}x, K^{-1}x) + \cdots + (K^{-n}x, K^{-n+1}x)$$

are the norms of the energetic Frechet space $E_{K^{-\infty}}$ of the operator $K^{-\infty}$. Thus, in the space $D(K^{-\infty})$ there exists two sequences of norms $\|f\|_n$ and $[f]_n$. From (4) follows that the second sequence generates a stronger topology. Let $\widetilde{(X, \|\cdot\|)}$ denote a completion of a normed space $(X, \|\cdot\|)$. It is well-known ([4], p.76), that the map of $\widetilde{(D(K^{-\infty}), [\cdot]_n)}$ into $\widetilde{(D(K^{-\infty}), \|\cdot\|_n)}$ is one-to-one and continuous. From this we conclude that these sequences of norms generate comparable topologies on the Frechet space $D(K^{-\infty})$ and hence they coincide.

It is clear that a basic sequence $\varphi_k \in D(K^{-\infty})$, $k \in N$, is a sequence of eigenfunctions of $K^{-\infty}$ too. Therefore, the operator $K^{-\infty}$ has the everywhere dense range in the Frechet space $D(K^{-\infty})$. By ([2], Theorem 1) there exists the inverse operator $(K^{-\infty})^{-1}$ of $K^{-\infty}$. It is continuous since the operator $K^{-\infty}$ is positive definite and selfadjoint along with $K^{-\infty}$. Therefore the operator $K^{-\infty}$ is an isomorphism of $D(K^{-\infty})$ onto itself and the equation $(K^{-\infty})^{-1}u = f$ has in the space $D(K^{-\infty})$ a unique and stable solution. Let us denote the operator $(K^{-\infty})^{-1}$ by K_∞. This operator K_∞ coincides with the restriction of the operator K^N from the space H^N on the Frechet space $D(K^{-\infty})$. Noting that $K_\infty u = \{Ku, u, K^{-1}u, \cdots, K^{-n+2}u, \cdots\}$, we have

$$K^{-\infty}K_\infty u = K_\infty K^{-\infty}u = \{u, K^{-1}u, \cdots, K^{-n}u\} = u.$$

Similarly to [1], we transfer the equation $Ku = f$ to the metric Frechet space $E = D(K^{-\infty})$, on which the restriction K_∞ of the operator K is a selfadjoint operator. Moreover, K_∞ is onto isomorphism of $D(K^{-\infty})$ and therefore the equation $K_\infty u = f$ has in $D(K^{-\infty})$ a unique and stable solution. As a set, $D(K^{-\infty})$ is a part of H and the restriction of K on the space $D(K^{-\infty})$, taking into account the topology, by (3) is factually a restriction of the operator K^N from H^N onto $D(K^{-\infty})$.

Based on the described above we consider the equation $K_\infty u = f$ instead of $Ku = f$. For its approximate solution in the Frechet space $D(K^{-\infty})$ we apply Ritz's extended method for only positive operators. This is possible because of a remark which is given in ([4], $\S 4, p.193$). To this end we consider the energetic Frechet space E_{K_∞} of the operator K_∞, the energetic norms of which has the form

$$[x]_n^{'} = (K_\infty x, x)^{1/2} = ((Kx, x) + (KK^{-1}x, K^{-1}x) + \cdots + (K^{-n+2}x, K^{-n+1}x))^{1/2}\ .$$

As a basic functions we take a sequence $\{\varphi_k\}$ of eigenfunctions of K. The system for the coefficients of the approximate solution $u_m = \sum_{k=1}^{m} a_k\varphi_k$ by the Ritz's extended method has the following form

$$\sum_{k=1}^{m} a_i[\varphi_k, \varphi_i]_l^{'} = (f, \varphi_k)_l\ ,$$

where $[\varphi_k, \varphi_i]_l^{'} = (K_\infty\varphi_k, \varphi_i)_l^{1/2}$. By calculation we have $a_k = (f, \varphi_k)/((\varphi_k, \varphi_k)\sigma_k)$ and hence the approximate solutions has the form

$$u_m = \sum_{k=1}^{m} (f, \varphi_k)/((\varphi_k, \varphi_k)\sigma_k)\varphi_k\ .$$

Thus the approximate solution again does not depend on l. This means that the subspace G_m, which is spanned on the functions $\varphi_1, \varphi_2, \cdots, \varphi_m$, has an orthogonal complement in the Frechet space E_{K_∞} [2]. Let us prove that the sequence of the approximate solutions converges to the element $(K_\infty)^{-1}f$ in the Frechet space E_{K_∞}. Let us define the canonical maps $K_n : E_{K_\infty} \to E_{K_\infty}/Ker[\cdot]_n^{'}$, being identical maps J_n defined by equalities $J_n x = J_n\{x, K^{-1}x, \cdots, K^{-n+1}x, \cdots\} = \{x, K^{-1}x, \cdots, K^{-n+1}x)\}$. The projection operators of K_∞ has the form

$$K_{\infty,n}(J_n(x)) = J_n(K_\infty(x)) = \{Kx, x, K^{-1}x, \cdots, K^{-n+2}x\}.$$

They map the Hilbert space $(\widetilde{D(K^{-\infty})}, [\cdot]_n^{\prime})$ into. In the space $(\widetilde{D(K^{-\infty})}, [\cdot]_n^{\prime})$ let us define the energetic space $(\widetilde{D(K^{-\infty})}, [\cdot]_n^{\prime})_{K_{\infty,n}}$ of the operator $K_{\infty,n}$ and the energetic norm by the equality

$$[J_n x]_{K_\infty,n} =< K_{\infty,n}J_n x, J_n x >= ((Kx,x) + \cdots + (K^{-n+2}x, K^{-n+1}x))^{1/2},\ n \in N.$$

The norms $[\cdot]_n^{'}$ and $[\cdot]_{K_\infty,n}$ and hence the spaces $(\widetilde{D(K^{-\infty})}, [\cdot]_n^{\prime})_{K_{\infty,n}}$ and $(\widetilde{D(K^{-\infty})}, [\cdot]_n^{\prime})$ are isometric for any $n \in N$. According to Ritz's classical theorem (for only positive operators) we obtain that the sequence $\{u_m\}$ of approximate solutions converges to the element $(K_\infty)^{-1}f$ in the norm $[\cdot]_{K_\infty,n}$ in $(\widetilde{D(K^{-\infty})}, [\cdot]_n^{\prime})_{K_{\infty,n}}$ for every $n \in N$. Because of norms isometricity this sequence converges in E_{K_∞} too. On the other hand, we have for any $n \in N$ the inequalities $[f]_n \leq [f]_{n+1}^{'}$. This means that the norm sequence $\{[\cdot]_n^{'}\}$ define on the space $D(K^{-\infty})$ a stronger topology than $\{[\cdot]_n\}$. This implies the following assertion.

Theorem. Let $K : H \to H$ be a compact selfadjoint one-to-one operator in a Hilbert space H with positive eigenvalues with an everywhere dense range. Then the sequence of approximate solutions for the equation $K_\infty u = f$ constructed by Ritz's extended method for the sequence of eigenfunctions $\{\varphi_k\}$, converges to the stable solution $(K_\infty)^{-1}f$ in both the energetic space E_{K_∞} of operator K_∞ and in the Frechet space $D(K^{-\infty})$.

Let us give examples of selfadjoint operators for which the conditions of Theorem are satisfied.

In [2] there is considered the selfadjoint and positive definite harmonic oscillator operator $A(u) = -u^{''} + t^2 u$, $A : L^2(R) \to L^2(R)$. It is well known, that the eigenfunctions φ_k of this operator are wave functions and the correspondent eigenvalues equal to $2k + 1$. Therefore, for the operator $K = A^{-1}$, having the form $K(u) = \sum_{k=1}^{\infty}(u, \varphi_k)/(2k+1)\varphi_k$ all conditions of Theorem are fulfilled. Moreover, we have the equalities $D(K^{-\infty}) = D(A^\infty) = S(R)$ [2],where $S(R)$ is a well-known Schwartz space. In this case the sequence of functions $\{\varphi_k\}$ is orthonormal and the approximate solutions has the form $u_m = \sum_{k=1}^{m}(f, \varphi_k)(2k+1)\varphi_k$.

As other examples one may use inverse operators of strongly degenerate elliptic operators of [2]. In particular, such operators are the inverse operators for Lagrange, Legandre and Tricomi operators,with a discrete spectrum.

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