Reports of Enlarged Session of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 21, 2006-2007

THE GREEN'S FUNCTION OF THE INFINITY BOUNDARY VALUE PROBLEM FOR THE EQUATION OF THE RADIATION TRANSPORT THEORY

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The penetration of radiation, primarily gamma rays and neutrons, through thick reactor shields obeys a linear transport equation [1]. The our aim is to construct the Green's function of the infinity problem by applying the eigenfunction expansions method [2]. The discussion will be limited in this paper to the penetration through infinite homogeneous isotropic media in plane geometry. Consider the following equation of radiation transport

$$\mu\ell\frac{\partial\Psi}{\partial x}+\Psi=\int_{\lambda_0}^{\lambda}\int_{-1}^{+1}K\Psi d\mu' d\lambda', \tag{1}$$

$$x\in(-\infty,+\infty),\quad \mu\in(-1,+1),\quad \lambda\in[\lambda_0,\lambda_1],$$

where $\ell(\lambda)>0$ is the continuous function, the function $K(\mu,\lambda,\lambda',\mu')$ is continuous and satisfy the H (Hölder) conditions [3] with respect to μ,μ'.

As is known [2], every continuous solution $\Psi(x,\mu,\lambda)$ of the equation (1) differentiable with respect to x, satisfying the condition H^* with respect to μ admits the following representation

$$\Psi=\int_{-l_{\max}}^{+l_{\max}}\int_{m(\nu)}\exp\frac{x-x_0}{\nu}\varphi_{\nu,(\zeta)}u(\nu,\zeta)d\nu d\zeta,$$

where $l_{\max}=\max l(\lambda)$, x_0 is a constant, u is a continuous function,

$$\varphi_{\nu,(\zeta)}=\frac{\nu M(\nu,\zeta;\mu,\lambda)}{\nu-\mu l(\lambda)}$$

$$+(\delta(\zeta-\lambda)-\int_{-1}^{+1}\frac{\nu M(\nu,\zeta;\mu',\lambda)}{\nu-\mu' l(\lambda)}d\mu')\delta(\nu-\mu l(\lambda)$$

is the singular eigenfunction of the characteristic equation

$$(\nu-\mu l(\lambda))\varphi_\nu=\nu\int_{\lambda_0}^{\lambda}\int_{-1}^{+1}K(\mu,\lambda,\mu',\lambda')\varphi_\nu d\mu' d\lambda'.$$

Here $M(\nu,\zeta;\mu,\lambda)$ is the unique solution of the following second kind regular integral equation

$$M(\nu,\zeta;\mu,\lambda)=K(\mu,\lambda,\nu l^{-1}(\zeta),\zeta)$$

$$+\int_{\lambda_0}^{\lambda}\int_{-1}^{+1}\frac{K(\mu,\lambda,\mu',\lambda')-\theta(l(\lambda')-\mid\nu\mid))K(\mu,\lambda,\nu l^{-1}(\lambda'),\lambda')}{\nu-\mu l(\lambda')}M(\nu,\zeta;\mu',\lambda')d\mu'd\lambda',$$

$$\nu\in[-l_{max},+l_{max}],\quad \zeta\in m(\nu),\quad \mu\in[-1,+1],\quad \lambda\in[\lambda_0,\lambda_1],$$

$\theta(t)$ is the Heaviside function, $\quad m(\nu)=\{\zeta\in[\lambda_0,\lambda_1]:l(\zeta)\geq\mid\nu\mid\}$.

There are not regular eigenfunctions. Let $\varphi^*_{\nu,(\zeta)}$ be the singular eigenfunction of the conjugate equation

$$(\nu-\mu l(\lambda))\varphi^*_\nu=\nu\int_{\lambda}^{\lambda_1}\int_{-1}^{+1}K(\mu'.\lambda',\mu,\lambda)\varphi^*_\nu d\mu'd\lambda'$$

and

$$\bar{\varphi}^*_{\nu,(\zeta)}(\mu,\lambda)=\varphi^*_{\nu,(\zeta)}(\mu,\lambda)+\int_{m(\nu)}r(\nu,\zeta,\zeta')\varphi^*_{\nu,(\zeta')}(\mu,\lambda)d\zeta',$$

where $r(\nu,\zeta,\zeta')$ unique solution of the following regular equation

$$r(\nu,\zeta,\zeta')-\int_{m(\nu)}g(\nu,\zeta'',\zeta')r(\nu,\zeta,\zeta'')d\zeta''=g(\nu,\zeta,\zeta'),$$

$$g(\nu,\zeta,\zeta')=-\pi^2\nu^2\int_{m(\nu)}M(\nu,\zeta',\nu l^{-1}(\lambda'),\lambda')M^*(\nu,\zeta,\nu l^{-1}(\lambda'),\lambda')d\lambda'$$

$$+\int_{-1}^{+1}\frac{\nu M(\nu,\zeta';\mu,\zeta)}{\nu-\mu l(\zeta)}d\mu+\int_{-1}^{+1}\frac{\nu M^*(\nu,\zeta;\mu,\zeta')}{\nu-\mu l(\zeta')}d\mu$$

$$-\int_{m(\nu)}\int_{-1}^{+1}\frac{\nu M(\nu,\zeta';\mu,\lambda')}{\nu-\mu l(\lambda')}d\mu\int_{-1}^{+1}\frac{\nu M^*(\nu,\zeta;\mu,\lambda')}{\nu-\mu l(\lambda')}d\mu d\lambda',$$

$$\nu\in(-l_{max},l_{max}),\quad \zeta,\zeta'\in m(\nu),$$

$M^*(\nu,\zeta;\mu,\lambda)$ is the solution of the equation

$$M^*(\nu,\zeta;\mu,\lambda)=K(\nu l^{-1}(\zeta),\zeta,\mu,\lambda)$$

$$+\int_{\lambda}^{\lambda_1}\int_{-1}^{+1}\frac{K(\mu',\lambda',\mu,\lambda)-\theta(l(\lambda')-\mid\nu\mid))K(\nu l^{-1}(\lambda'),\lambda';\mu,\lambda)}{\nu-\mu l(\lambda')}M^*(\nu,\zeta;\mu',\lambda')d\mu'd\lambda',$$

$\zeta\in m(\nu)$ is the parameter.

The system of singular eigenfunctions is complete and the following equality

$$\mu l(\lambda)\int_{-l_{\max}}^{+l_{\max}}\int_{m(\nu)}\varphi_{\nu,(\zeta)}(\mu,\lambda)\bar{\varphi}^*_{\nu,(\zeta)}(\mu',\lambda')d\nu d\zeta=\delta(\mu-\mu')\delta(\lambda-\lambda') \tag{2}$$

holds.

The systems of eigenfunctions represents the orthonormal systems, i.e. the equality

$$\int_{\lambda_0}^{\lambda_1}\int_{-1}^{+1}\mu l(\lambda)\varphi_{\nu,(\zeta)}\bar{\varphi}^*_{\nu',(\zeta')}d\mu d\lambda=\delta(\nu-\nu')\delta(\zeta-\zeta') \tag{3}$$

holds.

Now we able to construct the Green's function of infinity boundary value problems.

1. The Green's function for infinity domain it is defined as follows. Is required to find vanishing in infinity continuous in $(-\infty, x_0) \cup (x_0, +\infty)$ the solution Ψ_1 of (1) satisfying the condition

$$\mu l(\lambda)(\Psi_1(x_0^+, \mu, \lambda) - \Psi_1(x_0^-, \mu, \lambda)) = \delta(\mu - \mu')\delta(\lambda - \lambda'),$$

where $x_0 \in (-\infty, +\infty)$, $\mu' \in (-1, +1)$ $\lambda' \in [\lambda_0, \lambda_1]$ are constants.

By the above mentioned formulas the unique solution Ψ_1 of the formulated problem can be represent in the form

$$\Psi_1(x, \mu, \lambda)$$
$$= \pm \int_0^{+l\max} \int_{m(\nu)} \exp \frac{(x - x_0)}{\pm\nu} \varphi_{\pm\nu,(\zeta)}(\mu, \lambda) \bar{\varphi}^*_{\pm\nu,(\zeta)}(\mu', \lambda') d\nu d\zeta,$$

where upper signs taken when $x > x_0$, and lower signs taken when $x < x_0$.

2. The Green's Function for two Half-Space it is defined as follows. Is required to find vanishing when $x \to -\infty$ continuous in $x < 0$ the solution Ψ_2 of the (1), in other half-space $x > 0$ vanishing when $x \to +\infty$ the continuous solution of the (1) in $(0, x_0) \cup (x_0, +\infty)$ satisfying the condition

$$\mu l(\lambda)(\Psi_2(x_0^+, \mu, \lambda) - \Psi_2(x_0^-, \mu, \lambda)) = \delta(\mu - \mu')\delta(\lambda - \lambda'),$$

(here $x_0 > 0$) and on interface $x = 0$ satisfying the condition

$$\Psi_2(0^+, \mu, \lambda) = \Psi_2(0^-, \mu, \lambda). \tag{4}$$

Continuous, vanishing in infinity the solution of (1), when $x < 0$ may be represent, in general, of the form

$$\Psi_2(x, \mu, \lambda) = \int_{-l_{\max}}^{0} \int_{m(\nu)} \exp \frac{x}{\nu} \varphi_{\nu,(\zeta)}(\mu, \lambda) u(\nu, \zeta) d\nu d\zeta,$$

and vanishing in infinity the solution of (1) when $x > 0$ may be represent, in general, of the form

$$\Psi_2(x, \mu, \lambda) = \Psi_1(x_0, \mu, \lambda)$$
$$- \int_0^{+l_{\max}} \int_{m(\nu)} \exp \frac{(x - x_0)}{\nu} \varphi_{\nu,(\zeta)}(\mu, \lambda) u(\nu, \zeta) d\nu d\zeta.$$

In view of (4) we have

$$\Psi_1(x_0, \mu, \lambda) = \int_{-l_{\max}}^{+l_{\max}} \int_{m(\nu)} \varphi_{\nu,(\zeta)}(\mu, \lambda) u(\nu, \zeta) d\nu d\zeta.$$

From this, to find u, we apply (3), then it follows that

$$u(\nu, \zeta) = \int_{\lambda_0}^{\lambda_1} \int_{-1}^{+1} \mu l(\lambda) \Psi_1 \bar{\varphi}^*_{\nu,(\zeta)} d\mu d\lambda.$$

R E F E R E N C E S

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Received 29.09.2006; revised 26.03.2007; accepted 25.12.2007.