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APPROXIMATION OF A LOCAL SOLUTION OF THE KIRCHHOFF STRING EQUATION

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In 1883, G. Kirchhoff obtained the equation of string vibration

$$w_{tt}(x,t)-\left(\lambda+\frac{2}{\pi}\int_0^\pi w_x^2(x,t)dx\right)w_{xx}(x,t)=0,\quad 0<x<\pi,\quad 0<t\le T. \tag{1}$$

Here $\lambda>0$ and T are the given constants. This equation is a particular case of the equation which was for the first time investigated by S. Bernstein [1] in 1940. In subsequent years, equation (1) and its various generalizations were studied in the works of A. Arosio, J.Ball, M.Bőhm, G.Carrier, P.D'Ancona, R.Dickey, W.Newman, K.Nishihara, S.Panizzi, S.Pohozaev, R.Rodriguez, S.Spagnolo and other authors. An overwhelming majority of the works were concerned with finding solvability conditions, while little attention was given to the construction of approximate algorithms and the establishment of their accuracy. Besides the works of F. Attigui, S. Bilbao, L. Liu, J. Peradze, M. Rincon, where this problem was to this extent or another touched upon, we hardly know of any other papers published in this direction.

Let us consider equation (1) with the initial boundary conditions:

$$w(x,0)=w^0(x),\ w_t(x,0)=w^1(x),\ w(0,t)=w(\pi,t)=0,\ 0\le x\le\pi,\ 0\le t\le T. \tag{2}$$

Let us assume that $w^0(x)$ and $w^1(x)$ are the given $2\pi-$periodic functions of the form:

$$\begin{gathered}w^l(x)=\sum_{i=1}^\infty a_i^{(l)}\sin ix,\quad l=0,1,\\ |a_i^{(0)}|\le\frac{\Omega}{i^{p+1.5}},\quad |a_i^{(1)}|\le\frac{\Omega}{i^{p+0.5}},\ i=1,2,\dots,\ p>2,\ \Omega>0.\end{gathered} \tag{3}$$

From [1] follows that if conditions (3) are fulfilled there exists a solution $w(x,t)$ of problem (1), (2) for $T<T_*$. Here

$$T_*=\left\{\lambda^{\frac12}\sum_{i=1}^\infty i\left[i^2a_i^{(0)2}+\left(\lambda+\sum_{j=1}^\infty j^2a_i^{(1)2}\right)^{-1}a_i^{(1)2}\right]\right\}^{-1}. \tag{4}$$

Thus we have a local solution. An approximate solution will be sought as a series $w_n(x,t)=\sum_{i=1}^n w_{ni}(t)\sin ix$, where the coefficients $w_{ni}(t)$ are defined by the Galerkin method from the system of nonlinear differential equations with the initial conditions:

$$w_{ni}''(t)+\left(\lambda+\sum_{j=1}^n j^2w_{nj}^2(t)\right)i^2w_{ni}(t)=0,\qquad 0<t\le T, \tag{5}$$

$$w_{ni}(0) = a_i^{(0)}, \qquad w'_{ni}(0) = a_i^{(1)}, \quad i = 1, 2, \ldots, n. \tag{6}$$

We introduce the functions $u_{ni}(t) = w'_{ni}(t)$, $v_{ni}(t) = iw_{ni}(t)$, $i = 1, 2, \ldots, n$, and replace system (5),(6) by an equivalent system whose operator form is

$$\boldsymbol{u}'_n(t) + (\lambda + ||\boldsymbol{v}_n(t)||_n^2)K_n\boldsymbol{v}_n(t) = 0, \quad \boldsymbol{v}'_n(t) = K_n\boldsymbol{u}_n(t), \qquad 0 < t \leq T, \tag{7}$$

$$\boldsymbol{u}_n(0) = \boldsymbol{a}_n^1, \quad \boldsymbol{v}_n(0) = K_n\boldsymbol{a}_n^0. \tag{8}$$

Here we use the notations $\boldsymbol{u}_n(t) = (u_{ni}(t))_{i=1}^n$, $\boldsymbol{v}_n(t) = (v_{ni}(t))_{i=1}^n$, $\boldsymbol{a}_n^j = (a_i^{(j)})_{i=1}^n$, $j = 0, 1$, $K_n = diag(1, 2, \ldots, n)$, while the norm $||\cdot||_n$ of a vector $v \in R^n$, $v = (v_i)_{i=1}^n$, is defined by the formula $||v||_n = \left(\sum_{i=1}^n v_i^2\right)^{\frac{1}{2}}$. Let us introduce some additional definitions. We construct the block vector $\boldsymbol{s}_n(t) = (\boldsymbol{u}_n(t), \boldsymbol{v}_n(t))$. Here and in what follows the sign of transposition of the vectors is omitted. Besides, we introduce the block matrices A_n and $B_n(s)$, $s = (u, v)$,

$$A_n = \begin{pmatrix} 0 & -\lambda K_n \\ K_n & 0 \end{pmatrix}, \quad B_n(s) = \begin{pmatrix} 0 & -||v||_n^2 K_n \\ 0 & 0 \end{pmatrix}$$

and the block vector $\boldsymbol{a}_n = (\boldsymbol{a}_n^1, K_n\boldsymbol{a}_n^0)$. Now we can write system (7), (8) in the form

$$\boldsymbol{s}'_n(t) = (A_n + B_n(\boldsymbol{s}_n(t)))\boldsymbol{s}_n(t), \quad 0 < t \leq T, \tag{9}$$

$$\boldsymbol{s}_n(0) = \boldsymbol{a}_n. \tag{10}$$

We will solve problem (9), (10) by means of a difference scheme. On the time interval $[0, T]$ we introduce the grid $\{t_m | 0 = t_0 < t_1 < \cdots < t_M = T\}$ with a generally variable step $\tau_m = t_m - -t_{m-1}$, $m = 1, 2, \ldots, M$. An approximate value of $\boldsymbol{s}_n(t)$ on the mth time level, i.e. for $t = t_m$, $m = 0, 1, \ldots, M$, denoted by the vector $\boldsymbol{s}_n^m = (\boldsymbol{u}_n^m, \boldsymbol{v}_n^m)$, $\boldsymbol{u}_n^m, \boldsymbol{v}_n^m \in R^n$, is found by a modification of the Crank-Nicolson scheme

$$\frac{\boldsymbol{s}_n^m - \boldsymbol{s}_n^{m-1}}{\tau_m} = \left[A_n + \frac{1}{2}(B_n(\boldsymbol{s}_n^m) + B_n(\boldsymbol{s}_n^{m-1}))\right]\frac{\boldsymbol{s}_n^m + \boldsymbol{s}_n^{m-1}}{2}, \; m = 1, 2, \ldots, M, \tag{11}$$

$$\boldsymbol{s}_n^0 = \boldsymbol{a}_n. \tag{12}$$

Let us consider the question of solving the system of nonlinear equations (11),(12). Note that each equation of (11) contains the vectors $\boldsymbol{s}_n^{m-l}$, $l = 0, 1$, from two time levels. It is assumed that the counting is performed levelwise, and one and the same iteration process is realized on each mth level. In equation (11) the vector $\boldsymbol{s}_n^{m-1}$ is replaced by the vector $\boldsymbol{s}_n^{m-1,F} = (\boldsymbol{u}_n^{m-1,F}, \boldsymbol{v}_n^{m-1,F})$, $\boldsymbol{u}_n^{m-1,F}$, $\boldsymbol{v}_n^{m-1,F} \in R^n$, which is the final ($F$) iteration approximation for $\boldsymbol{s}_n^{m-1}$ obtained on the $(m-1)$th level. Therefore the vector $\boldsymbol{s}_n^m$ cannot be found exactly. Instead of $\boldsymbol{s}_n^m$, the vector $\boldsymbol{s}_{n,R}^m = (\boldsymbol{u}_{n,R}^m, \boldsymbol{v}_{n,R}^m)$, $\boldsymbol{u}_{n,R}^m, \boldsymbol{v}_{n,R}^m \in R^n$, is a real ($R$) solution of the resulting equation. Thus the equation

$$\frac{\boldsymbol{s}_{n,R}^m - \boldsymbol{s}_n^{m-1,F}}{\tau_m} = \left[A_n + \frac{1}{2}(B_n(\boldsymbol{s}_{n,R}^m) + B_n(\boldsymbol{s}_n^{m-1,F}))\right]\frac{\boldsymbol{s}_{n,R}^m + \boldsymbol{s}_n^{m-1,F}}{2}$$

corresponds to the mth level, $m > 1$. Since starting from the second level, the same situation takes place on every level, it is natural, in the latter equation to replace $\boldsymbol{s}_n^{m-1,F}$ by

$$\boldsymbol{s}_{n,R}^{m-1,F} = (\boldsymbol{u}_{n,R}^{m-1,F}, \boldsymbol{v}_{n,R}^{m-1,F}), \tag{13}$$

$\boldsymbol{u}_{n,R}^{m-1,F}, \boldsymbol{v}_{n,R}^{m-1,F} \in R^n$. As a result, for $\boldsymbol{s}_{n,R}^m$ we have the equation

$$\frac{\boldsymbol{s}_{n,R}^m - \boldsymbol{s}_{n,R}^{m-1,F}}{\tau_m} = \left[A_n + \frac{1}{2}(B_n(\boldsymbol{s}_{n,R}^m) + B_n(\boldsymbol{s}_{n,R}^{m-1,F}))\right] \frac{\boldsymbol{s}_{n,R}^m + \boldsymbol{s}_{n,R}^{m-1,F}}{2}. \tag{14}$$

Not to introduce a special equation for the case $m = 1$, it is assumed that (14) is fulfilled for $m = 1$ as well, provided that $\boldsymbol{s}_{n,R}^{0,F}$ implies $\boldsymbol{s}_n^0$. To solve the nonlinear equation (14), we use a Picard type iteration process

$$\boldsymbol{s}_{n,R}^{m,k} = \boldsymbol{s}_{n,R}^{m-1,F} + \frac{\tau_m}{2}\left[A_n + \frac{1}{2}(B_n(\boldsymbol{s}_{n,R}^{m,k-1}) + B_n(\boldsymbol{s}_{n,R}^{m-1,F}))\right](\boldsymbol{s}_{n,R}^{m,k-1} + \boldsymbol{s}_{n,R}^{m-1,F}), \tag{15}$$

where

$$\boldsymbol{s}_{n,R}^{m,k-l} = (\boldsymbol{u}_{n,R}^{m,k-l}, \boldsymbol{v}_{n,R}^{m,k-l}), \tag{16}$$

$k = 1, 2 \ldots$ $\boldsymbol{u}_{n,R}^{m,k-l}$, $\boldsymbol{v}_{n,R}^{m,k-l} \in R^n$, is the $(k-l)$th iteration approximation of the vector $\boldsymbol{s}_{n,R}^m$, $l = 0, 1$, $\boldsymbol{s}_{n,R}^{m,0}$ is the initial approximation on the mth level. Thus the approximation to $\boldsymbol{s}_m^n$ is performed by using vectors $\boldsymbol{s}_{n,R}^{m,k}$, $k = 0, 1, \ldots$. Let us write the iteration process (15) componentwise. To do so, first the vectors from (13) and (16) are represented as $\boldsymbol{u}_{n,R}^{m-1,F} = (u_{ni,R}^{m-1,F})_{i=1}^n$, $\boldsymbol{v}_{n,R}^{m-1,F} = (v_{ni,R}^{m-1,F})_{i=1}^n$, $\boldsymbol{u}_{n,R}^{m,k-l} = (u_{ni,R}^{m,k-l})_{i=1}^n$, $\boldsymbol{v}_{n,R}^{m,k-l} = (v_{ni,R}^{m,k-l})_{i=1}^n$. We obtain

$$\begin{aligned} u_{ni,R}^{m,k} &= u_{ni,R}^{m-1,F} - \frac{\tau_m i}{2}\left\{\lambda + \frac{1}{2}\left[\sum_{j=1}^n \left((v_{nj,R}^{m,k-1})^2 + (v_{nj,R}^{m-1,F})^2\right)\right]\right\}(v_{ni,R}^{m,k-1} + v_{ni,R}^{m-1,F}), \\ v_{ni,R}^{m,k} &= v_{ni,R}^{m-1,F} + \frac{\tau_m i}{2}(u_{ni,R}^{m,k-1} + u_{ni,R}^{m-1,F}), \end{aligned} \tag{17}$$

$$m = 1, 2, \ldots, M, \quad k = 1, 2, \ldots, \quad i = 1, 2, \ldots, n.$$

We calculate the components $u_{ni,R}^{m,k}$ and $v_{ni,R}^{m,k}$ by formulas (17). Then, for chosen n and for $t = t_m$, the series $\sum_{i=1}^n \frac{1}{i} v_{ni,R}^{m,k} \sin ix$, gives, at the kth iteration step, an approximation value of the exact solution $w(x, t_m)$ of problem (1), (2). We can characterize the error of the algorithm by $\Delta w_{n,R}^{m,k}(x) = w(x, t_m) - \sum_{i=1}^n \frac{1}{i} v_{ni,R}^{m,k} \sin ix$. Let us denote

$$\omega^0 = \left[\frac{4}{\pi}\int_0^\pi (w^1(x))^2 dx + \left(\lambda + \frac{2}{\pi}\int_0^\pi \left(\frac{dw^0}{dx}(x)\right)^2 dx\right)^2\right]^{\frac{1}{2}} - \lambda,$$

$$\omega_{n,R}^{m-1,F} = \left[2\|\boldsymbol{u}_{n,R}^{m-1,F}\|_n^2 + (\lambda + \|\boldsymbol{v}_{n,R}^{m-1,F}\|_n^2)^2\right]^{\frac{1}{2}} - \lambda,$$

$$h_m = \frac{1}{2}\Big[\max(1,\lambda)\|\boldsymbol{s}_{n,R}^{m,0} + \boldsymbol{s}_{n,R}^{m-1,F}\|_n + \frac{1}{2}(\|\boldsymbol{v}_{n,R}^{m,0}\|_n^2 + \|\boldsymbol{v}_{n,R}^{m-1,F}\|_n^2)\|\boldsymbol{v}_{n,R}^{m,0} + \boldsymbol{v}_{n,R}^{m-1,F}\|_n\Big].$$

Let m_F be the number of iterations performed on the mth level.

Theorem. *Assume that conditions* (3) *are fulfilled, thereby ensuring the existence of a local solution of problem* (1), (2), *i.e. of a solution for* $T < T_*$, *where* T_* *is defined by formula* (4). *Choose a value* σ *such that* $0 < \sigma < 1$. *Assume that on each* m*th level,* $m = 1, 2, \ldots, m_0$, $1 \le m_0 \le M$, *the parameter* q_m *is such that* $0 < q_m < 1$, *and the step* τ_m *satisfies the inequalities*

$$\tau_m < \frac{2(1-\sigma)}{n}\Big(\max(1,\lambda) + 3\max(\omega^0, \omega_{n,R}^{m-1,F})\Big)^{-1},$$

$$\tau_m \le \frac{2q_m}{n}\Big\{\max(1,\lambda) + \frac{1}{3}\|\boldsymbol{v}_{n,R}^{m-1,F}\|_n^2 + \frac{3}{2}\Big[\frac{1}{3}(\omega_{n,R}^{m-1,F})^{\frac{1}{2}} +$$

$$+ \max(\|\boldsymbol{s}_{n,R}^{m,0}\|_n, \|\boldsymbol{s}_{n,R}^{m-1,F}\|_n + \tau_m n h_m) + (\|\boldsymbol{s}_{n,R}^{m,0} - \boldsymbol{s}_{n,R}^{m-1,F}\|_n + \tau_m n h_m)\frac{q_m}{1-q_m}\Big]^2\Big\}^{-1}.$$

Then, with chosen n *and* $t = t_{m_0}$, *the total error of the algorithm at the* k*th iteration step is estimated by*

$$||\Delta w_{n,R}^{m_0,k}(x)||_{L^2(0,\pi)} \le \sum_{l=1}^{2} C_{2l-1}\left(\frac{1}{n^p}\right)^{3-l} + C_2 e^{\gamma n}\max_{1\le m\le m_0}\tau_m^2 + \sum_{m=1}^{m_0-1} D_m q_m^{m_F} + D_{m_0} q_{m_0}^k,$$

where C_l, D_m, $l = 1, 2, 3$, $m = 1, 2, \ldots, m_0$, *and* γ *are the constants independent of* n, τ_m *and* k.

The case of a global solution is studied in [2].

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