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ON A DIFFERENCE SCHEME FOR SOLUTION OF SYSTEM OF EQUATIONS OF DYNAMICAL MULTIDIMENSIONAL PROBLEMS OF THE ELASTICITY THEORY AND SHELL THEORY

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At first we consider the problem for the equations of elasticity theory. Let $\overline{Q}_T = \overline{G} \times [0,T]$, $Q_T = G \times (0,T)$, where $\overline{G} = \{0 \le x_j \le \ell_\alpha,\ \alpha = 1,2,\dots,p\}$, be the p-dimensional parallelepiped.

We consider the problem of finding of solution for the system

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} = L\mathbf{u} + \mathbf{f}(\mathbf{x},\mathbf{t}), \quad (\mathbf{x},\mathbf{t}) \in \mathbf{Q_T}, \tag{1}$$

which satisfied the following complementary conditions

$$\mathbf{u} = \mathbf{0},\ \text{if } \mathbf{x} \in \mathbf{\Gamma} = \partial\mathbf{G},\ \mathbf{t} \in [\mathbf{0},\mathbf{T}],$$

$$\mathbf{u}(\mathbf{x},\mathbf{0}) = \mathbf{u_0}(\mathbf{x}),\ \frac{\partial \mathbf{u}(\mathbf{x},\mathbf{0})}{\partial \mathbf{t}} = \overline{\mathbf{u}}_\mathbf{0}(\mathbf{x}) \text{ if } \mathbf{x} \in \overline{\mathbf{G}}, \tag{2}$$

where

$$L\mathbf{u} = \mu\mathbf{\Delta u} + (\lambda+\mu)\mathbf{graddivu}, \quad \mathbf{\Delta u} = \sum_{\alpha=\mathbf{1}}^{\mathbf{p}} \frac{\partial^\mathbf{2}\mathbf{u}}{\partial \mathbf{x}_\alpha^\mathbf{2}},$$

$\lambda = const > 0$, $\mu = const > 0$ are the Lame coefficients, $\mathbf{u} = (\mathbf{u_1},\mathbf{u_2},\dots,\mathbf{u_n})$ is n-dimensional vector-function. The condition of positive definiteness is fulfilled for $c_1 = \mu$, $c_2 = \lambda + 2\mu$ (see [1]).

Let $\overline{\omega}_h = \{x_i = (i_1h_1,\dots,i_ph_p)\}$ be a net in $\overline{G}$, $0 \le i_j \le N_\alpha$, $h_\alpha = \ell_\alpha/N_\alpha$, $\alpha = 1,2,\dots,p$ and $\overline{\omega}_\tau = \{t_j = j\tau,\ j = 0,1,\dots\}$ be a net on the interval $0 \le t \le T$.

The set of the points (t_j,x_i), $x_i \in \overline{\omega}_h(\omega_h)$ and $t_j \in \omega_\tau$ is denoted by $\overline{G}_{h\tau}(G_{h\tau})$.

For net functions and their difference derivatives which are defined on $\overline{G}_{h\tau}$, we take the following notations: $y^{j+1}(x) = y(t_{j+1},x_i) = y((j+1)\tau,x_1,\dots,x_p)$, $\widehat{y} = y(t_{j+2},x_i)$, $\check{y} = y(t_j,x_i)$, $y_{\overline{t}} = (y-\check{y})/\tau$, $y_t = (\widehat{y}-y)/\tau$, $y^{(-1)} = y(t_{j+1},x_1,\dots,x_i-h_i,\dots,x_p)$, $y^{(+1)} = y(t_{j+1},x_1,\dots,x_i+h_i,\dots,x_p)$, $y_{\overline{x}_i} = (y-y^{(-1)})/h_i$, $y_{x_i} = (y^{(+1)}-y)/h_i$, $y_{\overset{\circ}{x}_i} = (y^{(+1)}-y^{(-1)})/2h_i = \frac{1}{2}(y_{x_i}+y_{\overline{x}_i})$.

Let us introduce a space H of net functions defined on ω_h and vanishing on Γ_h with the inner product $(\mathbf{y},\mathbf{v}) = \sum_{\mathbf{i}=\mathbf{1}}^{\mathbf{n}}(\mathbf{y_i},\mathbf{v_i})$,

$$(y_i,v_i) = \sum_{x\in\omega_h} y_i(x)v_i(x)h_1\dots h_p = \sum_{i_1=1}^{N_1-1}\sum_{i_2=1}^{N_2-1}\cdots\sum_{i_p=1}^{N_p-1} y_i(i_1h_1,\dots,i_ph_p)$$

$$\cdot v_i(i_1h_1,\ldots,i_ph_p)h_1\ldots h_p$$

and the norm $||\mathbf{y}|| = \sqrt{(\mathbf{y},\mathbf{y})}$. Besides we introduce the notations $A = A_1 + \cdots + A_p$, $A_i = -\Delta_{ii}$, $\Delta_{ii}\mathbf{y} = \mathbf{y_{x_i\overline{x}_i}}$.

The operator A is selfadjoint and positive in H. The norm in the energetic space H_A has the form

$$||\mathbf{y}||^2_{\mathbf{A}} = \sum_{\mathbf{i_1=1}}^{\mathbf{N_1}}\sum_{\mathbf{i_2=1}}^{\mathbf{N_2-1}}\cdots\sum_{\mathbf{i_p=1}}^{\mathbf{N_p-1}}\mathbf{y^2_{\overline{x}_1}h_1\ldots h_p} + \cdots + \sum_{\mathbf{i_1=1}}^{\mathbf{N_1-1}}\sum_{\mathbf{i_2=1}}^{\mathbf{N_2-1}}\cdots\sum_{\mathbf{i_p=1}}^{\mathbf{N_p}}\mathbf{y^2_{\overline{x}_p}h_1\ldots h_p}$$

or

$$||\mathbf{y}||^2_{\mathbf{A}} = \sum_{\mathbf{i=1}}^{\mathbf{p}}||\mathbf{y_{\overline{x}_i}}||^2_{\mathbf{i}}.$$

By analogy with [2], we put to the problem (1), (2) in the contrespondence the difference problem

$$(E+\sigma\tau^2R^i_j)\mathbf{y_{jt\overline{t}}} = \mathbf{L^i_{jh}y + f_j},\quad \mathbf{j = 1,2,\ldots,n}, \tag{3}$$

$$\mathbf{y = 0}\ \text{ if }\ \mathbf{x\in\Gamma_h},\ \mathbf{t}\in\overline{\omega}_\tau,\ \mathbf{y(x,0) = u_0(x)},\ \mathbf{y}_\tau\mathbf{(x,0)} = \widetilde{\mathbf{u}}_\mathbf{0}\mathbf{(x)}\ \text{ if }\ \mathbf{x}\in\overline{\omega}_\mathbf{h}, \tag{4}$$

where

$$L^i_{jh}\mathbf{y} = \sum_{\mathbf{j=1}}^{\mathbf{p}}\mathbf{a^i_{jj}y_{ix_i\overline{x}_j}} + \mathbf{2a}_{\alpha\beta}\sum_{\substack{\alpha,\beta=\mathbf{1}\\ \alpha\neq\beta}}^{\mathbf{p}}\boldsymbol{\Delta}_{\alpha\beta}\mathbf{y},$$

$$\Delta_{\alpha\beta}\mathbf{y} = \frac{\mathbf{1}}{\mathbf{4}}(\mathbf{y}_{\mathbf{x}_\alpha} + \mathbf{y}_{\overline{\mathbf{x}}_\alpha})_{\mathbf{x}_\beta} + \frac{\mathbf{1}}{\mathbf{4}}(\mathbf{y}_{\mathbf{x}_\alpha} + \mathbf{y}_{\overline{\mathbf{x}}_\alpha})_{\overline{\mathbf{x}}_\beta},\quad \alpha\neq\beta,$$

$$R^i_j = \frac{1}{2}a^i_{jj}A_j,\ a^i_{jj} = \lambda+2\mu,\ a^i_{jj} = \mu,\ i\neq j,\ a_{\alpha\beta} = \frac{1}{2}(\lambda+\mu),\quad i = 1,2,\ldots,p.$$

For example we write out the absolute approximating difference scheme for $p = 2$

$$(E+\sigma\tau^2R^i_j)\mathbf{y_{jt\overline{t}}} = \mathbf{L^i_{jh}y + f_j},\quad \mathbf{i,j = 1,2},$$

where
$L^1_{1h}\mathbf{y} = \mathbf{a^1_{11}y_{1x\overline{x}_1} + 2a_{12}\Delta_{12}y_2 + a^1_{22}y_{1x_2\overline{x}_2}}$, $L^2_{2h}\mathbf{y} = \mathbf{a^2_{22}y_{2x\overline{x}_2} + 2a_{12}\Delta_{12}y_1 + a^2_{11}y_{2x_1\overline{x}_1}}$, $\mathbf{a^1_{11} = a^2_{22}} = \lambda + \mathbf{2}\mu$, $a^2_{11} = a^1_{22} = \mu$, $a_{12} = \frac{1}{2}(\lambda+\mu)$, $\Delta_{12}\mathbf{y} = \frac{\mathbf{1}}{\mathbf{4}}(\mathbf{y_{x_1} + y_{\overline{x}_1}})_{\mathbf{x_2}} + \frac{\mathbf{1}}{\mathbf{4}}(\mathbf{y_{x_1}} + \mathbf{y_{\overline{x}_1}})\overline{\mathbf{x}}_\mathbf{2}$, $R^1_j = \frac{1}{2}a^1_{jj}A_j$, $R^2_j = \frac{1}{2}a^2_{jj}A_j$, $j = 1,2$.

For the difference schemes of a variant of shell theory (see [3]) we obtain a system of approximating equations of order N with $3N+3$ unknown functions. If we temporarily don't take into consideration the terms $\overset{k}{M}_j$ (see [3], ch. I, §10) which contain the unknown functions and their first order derivatives, then the resulting system of equations is subdivided into two groups: the first group contains the system of equations of plane problem of elasticity theory and the second — the Poisson equation of two space variables. Therefore, for them we write out the difference schemes analogously.

Further, the derivatives of first order are approximated by the central differences $y_{\overset{\circ}{x}_i}$.

The investigation of the received finite difference schemes is based on the general regularization principle (see [1], ch. VI, §3). The stability sufficient condition of the scheme (3) is given in the form

$$D > \frac{1}{4}\tau^2 L^i_{jh} \ \text{ or } \ D > \frac{1+\varepsilon}{4}\tau^2 L^i_{jh}, \tag{5}$$

where $D = E + \sigma\tau^2 R^i_j$ and L^i_{jh} are selfadjoint positive operators. Before to receiving of the stability sufficient conditions of the scheme (3), we write out some inequalities. Taking into account the conditions of ellipticity of operator L, it is possible to show that

$$c_1||\mathbf{y}||^{\mathbf{2}}_{\mathbf{A}} \leq (\mathbf{L^i_{jh}y}, \mathbf{y}) \leq \mathbf{c_2}||\mathbf{y}||^{\mathbf{2}}_{\mathbf{A}}.$$

Farther $\delta_1 E < A < \delta_2 E$, where E is identity operator

$$\delta_1 = \sum_{i=1}^{p} \frac{4}{h_i^2} \sin^2 \frac{\pi h_i}{2\ell_i}, \quad \delta_2 = \sum_{i=1}^{p} \frac{4}{h_i^2} \cos^2 \frac{\pi h_i}{2\ell_i}.$$

Hence we obtain $c_1\delta_1 E \leq L^i_{jh} \leq c_2\delta_2 E$. Then, it is possible make to use the inequalities

$$D > \frac{1}{4}\tau^2 c_2 A \ \text{ or } \ D \geq \frac{1}{4}\tau^2 c_2 \delta_2 E,$$

instead (5). From this we obtain the following sufficient condition of stability:

$$\frac{\tau}{h} \leq \frac{1}{\sqrt{c_2(p-2\sigma)}}, \quad 0 < \sigma \leq \frac{1}{2},$$

$$\frac{\tau}{h} < \frac{1}{\sqrt{c_2(p-1)}}, \quad \sigma > \frac{1}{2},$$

$$(h = h_1 = \cdots = h_p, \quad l_i = 1, \quad i = 1, 2, \ldots, p).$$

Remark. In the interval $\dfrac{1}{\sqrt{c_2 p}} < \dfrac{\tau}{h} \leq \dfrac{1}{\sqrt{c_2(p-2\sigma)}}$, $\left(0 < \sigma \leq \dfrac{1}{2}\right)$, where the stability conditions of explicit scheme are not safisfied in the family of scheme sets with the approximation $O(\tau^2 + |h|^2)$, the represented types of schemes have preferencies compared with other economical schemes.

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