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ON NUMERICAL SCHEMES OF THE NONLINEAR PARABOLIC TYPE EQUATION DESCRIBING FLUID FILTRATION IN SOIL

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Investigation of water soil products regime of soil is difficult problem. Such kind process as is penetration of moisture and oil products, exchange of mass between solution, exchange between gas-image and hard particles of soil is determined by influence of about 40 factors. It is necessary to have into account compound structure soil, influence of logical components of soil, transfer of impulse of electrical, magnetic and thermal fields of soils. That is why building of mathematical model which will take into account all real processes happening in soil is practically impossible.

Generally in the normal state the Earth soil represents a three-phase system-solid particles (mineral and colloidal particles), water (with salts dissolved in it) and air (air and water vapor). In the case when a big amount of oil is spilled on the Earth surface (at different emergencies of oil pipelines, railway accidents and so on), then soil represents a four-phase system-soil particles, water, oil and air with vapor of water and oil.

As known, the Earth soil contains two kinds of pore-capillary and non-capillary, which differ by scales and ability of oil-water filtration [1]. Capillary pores condition keeping of water and oil, and non-capillary pores contribute to quick seepage of liquid into the soil. As we mainly are interested in the subject of liquid filtration into the soil, therefore we will limit to the case, when in the soil exist only non-capillary pores. For this case we consider an equation for a saturation of soil by the liquid.

At the present time numerous of works devoted to the investigation of the problem of oil filtration to the soil [1-5]. Accounting to [1-4] the process of oil-products spreading in soil can be described by the equation(1).

In [5,6] are considered method of constractue as conditionally stable as well unconditionally stable difference schemes for linear parabolic type equation. In present article is constructed various two layers absolute stable difference schemes (similar analogous) to [5.6] solution of which are convergence to the solution of differential equations with the speed as $O(\tau + h^2)$ as $O(|\tau|^2 + |h|^2)$, where τ and h- are steps along time by and spickal axis $(h = (h_1, h_2))$.

1. Set of the problem. Let us $\overline{\Omega} = \Omega + \Gamma$ is two dimensional area with a bound Γ.

$$\Omega = \{0 < x_1 < L,\ 0 < x_2 < H\}.$$

We are finding for continous (uninteverhed) function $W(x,t)$, $x = (x_1, x_2) \in \overline{\Omega}$, which satisfy the following equation in the rectangular parallelepiped. : $\overline{Q}_T =$

$\overline{\Omega} \times [0 \leq t \leq T]$

$$\frac{\partial W}{\partial t} = \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(D(t,x,W)\frac{\partial W}{\partial x_i}\right) - A\frac{\partial W^n}{\partial x_2} + f, \qquad x \in \Omega, \qquad t > 0 \tag{1}$$

and additional conditives on the Γ

$$W(x_1, 0, t) = 0, \qquad x_1 \notin (\alpha, \beta) \subset [0, L], \tag{2}$$

$$W(x_1, 0, t) = W_1, \qquad x_1 \notin (\alpha, \beta),$$

$$\left.\frac{\partial W(x,t)}{\partial x_2}\right|_{x_2=H} = 0, \qquad \left.\frac{\partial W(x,t)}{\partial x_1}\right|_{x_1=0} = 0, \qquad \left.\frac{\partial W(x,t)}{\partial x_1}\right|_{x_2=L} = 0,$$

$$W(x, 0) = W_0, \qquad 0 \leq x_1 \leq L, \qquad 0 \leq x_2 \leq H, \tag{3}$$

where $D(t,x,W) \geq \alpha_1 = const,\ f, W_0, A$ are known functions. $n-$ is positive rational quartity (number). It is acceptable the interval (α, β) is contaius only one one point. Also let us suppose, that derivative of function D is satisfied the some conditions that is requested in [7].

Let us introduce to the following grids

$$\overline{\omega}_h = \left\{x_1 = i_1 h_1, x_2 = i_2 h_2, i_1 = 0, 1, .., N_1 = \frac{L}{h_1}, i_2 = 0, 1, .., N_2 = \frac{H}{h_2}, \right\}$$

and

$$\overline{\omega}_\tau = \{t_j = j\tau,\ j = 0, 1, ...j_0, j_0\tau = T\}$$

in the areas $\overline{\Omega}$ and $[0, T]$

For the grid functions and their difference derivatives which are given on the grid Let us introduce the following notatious :

$$\widehat{y} = y(t_{j+1}, x), \quad y = y(t_{j-1}, x), \qquad \overset{\vee}{y} = (t_{j-1}, x), \qquad y_{\overset{0}{t}} = \frac{1}{2}\left(y_t + y_{\overline{t}}\right),$$

$$y_t = (\widehat{y} - y)/\tau, \qquad y_{\overline{t}} = (y - \overset{\vee}{y})/\tau, \qquad y_{\overline{x}_1} = \left(y(t_j, x) - y(t_j, x_1 - h, x_2)\right)/h_1,$$

$$y_{x_1} = \left(y(t_j, x_1 + h, x_2) - y(t_j, x)\right)/h_1, \qquad y_{\overset{0}{x}} = \frac{1}{2}\left(y_x + y_{\overline{x}}\right).$$

2. Difference scheme. For the purpose of constraction of difference schemes at the beginning for simplicity let us consider one dimensional following case :

$$\frac{\partial W}{\partial t} = \frac{\partial}{\partial x_i}\left(D\frac{\partial W}{\partial x}\right) + f, \quad (t, x) \in \Omega = \{0 \leq x \leq L,\ 0 \leq t \leq T\}. \tag{4}$$

Let us rewrite (4) in the following form.

$$\beta\left(\frac{\partial W}{\partial t}+D\frac{\partial^2 W}{\partial t^2}\right)=\beta D\left(\frac{\partial^2 W}{\partial t^2}+\frac{\partial^2 W}{\partial x^2}\right)+\beta\frac{\partial D}{\partial x}\frac{\partial W}{\partial x}+\beta f,\qquad \beta\neq 0. \tag{5}$$

If we use the methodic of constraction of difference proposed in [5-6], [8], for the equation (4) we obtain the following difference scheme

$$y_{\overset{0}{t}}+\tau^2 R y_{t\bar{t}}=\overset{\vee}{D}\, y_{x\bar{x}}+D_{\overset{0}{t}}y_{\overset{0}{t}}+f(x,t_j), \tag{6}$$

where $\overset{\vee}{D}=D(t,x,\overset{\vee}{w})$, $\qquad R=\frac{1}{2}\sigma\,\overset{\vee}{D}\,\Delta$, $\qquad \Delta y=-y_{x\bar{x}}$

For the equation (5) γ- weighty scheme has the following form

$$y_{\overset{0}{t}}+\tau^2 R y_{t\bar{t}}=\gamma\left(\overset{\vee}{D}\,\widehat{y}_{x\bar{x}}+\overset{\vee}{D}_{\overset{0}{x}}\,\widehat{y}_{\overset{0}{x}}\right)+(1-2\gamma)\left(\overset{\vee}{D}\,y_{x\bar{x}}+\overset{\vee}{D}_{\overset{0}{\backslash x}}\,y_{\overset{0}{x}}\right)+f$$

If we suppose that $\sigma=1-2\gamma$ and $y_{\overset{0}{t}}$ substitute by semisum $y_{\overset{0}{x}}=\frac{1}{2}\widehat{y}_{\overset{0}{x}}+\overset{\vee}{y}_{\overset{0}{x}}$then we obtain absolutaly stable $O(\tau^2+h^2)$ accuracy two dimensional difference scheme.

3. Difference scheme for the task (1) - (3). Let us use for the equation (1) the following form :

$$\beta\left(\frac{\partial W}{\partial t}+D\frac{\partial^2 W}{\partial t^2}\right)=\beta D\left(\frac{\partial^2 W}{\partial t^2}+D\frac{\partial^2 W}{\partial x_1^2}\right)_{\delta_1}+\beta\left(\frac{\partial D}{\partial x_1}\frac{\partial W}{\partial x_1}\right)_{\delta_1}+F_1,$$

$$\beta\left(\frac{\partial W}{\partial t}+D\frac{\partial^2 W}{\partial t^2}\right)=\beta D\left(\frac{\partial^2 W}{\partial t^2}+D\frac{\partial^2 W}{\partial x_2^2}\right)_{\delta_2}+\beta\left(\frac{\partial D}{\partial x_2}\frac{\partial W}{\partial x_2}\right)_{\delta_2}+F_2,$$

$$\beta\left(\frac{\partial W}{\partial t}+2D\frac{\partial^2 W}{\partial t^2}\right)=\beta D\sum_{i=1}^{2}\left(\frac{\partial^2 W}{\partial t^2}+\frac{\partial^2 W}{\partial x_i^2}\right)_{\delta i}+\beta D\sum_{i=1}^{2}\left(\frac{\partial D}{\partial x_i}\frac{\partial W}{\partial x_i}\right)_{\delta i}+F_{3,}$$

where $\delta_1,\delta_2$and represents rectangle cross sections along the $x_1 o x_2$ and $x_1 o t$coordinate axises, respectively and pass through centre of elementary cell (with dimensionals $(2h_1,2h_2,2\tau_1)$

$$F_1=\beta\frac{\partial}{\partial x_2}\left(D\frac{\partial W}{\partial x_2}\right)_{\delta_2}-\beta A\frac{\partial W^n}{\partial x_2}+\beta f,$$

$$F_2=\beta\frac{\partial}{\partial x_1}\left(D\frac{\partial W}{\partial x_1}\right)_{\delta_1}-\beta A\frac{\partial W^n}{\partial x_2}+\beta f,$$

$$F_3=-A\beta\frac{\partial W^n}{\partial x_2}+\beta f,$$

Analogous of (5) are obtained the following schemes:

$$y_{\overset{0}{t}}+\tau^2 R_1 y_{t\bar{t}}=\overset{\vee}{D}\, y_{x_1\bar{x}_1}+\overset{\vee}{D}_{\overset{0}{x}_1}\, y_{\overset{0}{x}_1}+\overset{\vee}{F}_1, \tag{7}$$

$$y_{\overset{0}{t}}+\tau^2 R_2 y_{t\bar{t}}=\overset{\vee}{D}\, y_{x_2\bar{x}_2}+\overset{\vee}{D}_{\overset{0}{x}_2}\, y_{\overset{0}{x}_2}+\overset{\vee}{F}_2, \tag{8}$$

$$y_{\overset{0}{t}}+\tau^2(R_1+R_2)y_{t\bar{t}}=\left(\overset{\vee}{D}\,y_{x_1\bar{x}_1}+y_{x_2\bar{x}_2}\right)+\left(\overset{\vee}{D}_{\overset{0}{x_1}}\,y_{\overset{0}{x_1}}+\overset{\vee}{D}_{\overset{0}{x_2}}\,y_{\overset{0}{x_2}}\right)+\overset{\vee}{F}_{3,} \tag{9}$$

where $R_i=\frac{1}{2}\sigma\overset{\vee}{D}\Delta_{ii}\qquad \Delta_{ii}y=-y_{x_i\bar{x}_i},\qquad i=1,2.$

$$\overset{\vee}{F}_1=\frac{1}{h_1}\left[\overset{\vee}{a}_{i,j+1}\frac{\overset{\vee}{y}_{i,j+1}-\overset{\vee}{y}_{i,j}}{h_2}-\overset{\vee}{a}_{i,j}\frac{\overset{\vee}{y}_{i,j}-\overset{\vee}{y}_{i,j-1}}{h_2}\right]-A\overset{\vee n}{y}_{\overset{0}{x_2}}+f,$$

$$\overset{\vee}{F}_2=\frac{1}{h_1}\left[\overset{\vee}{a}_{i+1,j}\frac{\overset{\vee}{y}_{i+1,j}-\overset{\vee}{y}_{i,j}}{h_1}-\overset{\vee}{a}_{i,j}\frac{\overset{\vee}{y}_{i,j}-\overset{\vee}{y}_{i-1,j}}{h_1}\right]-A\overset{\vee n}{y}_{\overset{0}{x_2}}+f,$$

$$\overset{\vee}{F}_3=-\frac{A}{2}\left(\overset{\vee n}{y}_{\overset{0}{x_2}}+\overset{\vee n}{y}_{\overset{0}{x_2}}\right)+f,(f=f(x,t_j)),$$

$$a_i(v)=a(v_{i-1},v_{i)},$$

$$a_i(v)=0,5(D(v_{i,j-1}+D(v_{ij})). \tag{10}$$

$$a_i(v)=D\left(\frac{v_{i-1}+v_i}{2}\right), \tag{11}$$

$$a_i(v)=D\left(\frac{2D(v_{i-1})D(v_i)}{D(v_{i-1})+D(v_i)}\right), \tag{12}$$

$$a_i=a_i(y),\qquad \overset{\vee}{a}_i=a_i(\overset{\vee}{y}).$$

If we write γ- weighty schemes for the schemes (7) - (9) and suppose that $\sigma=1-2\gamma$, and substitute $y_{\overset{0}{x}}$ by midramge $\qquad y_{\overset{0}{x}}=\frac{1}{2}\left(\widehat{y}_{\overset{0}{x}}+\overline{y}_{\overset{0}{x}}\right).$

then we obtain corresponding two dimensional difference schemes:

$$y_{\overset{0}{t}}-\frac{\overset{\vee}{D}}{2}\widehat{y}_{x_1\bar{x}_1}-\frac{\overset{\vee}{D}_{\overset{0}{x_1}}}{2}\widehat{y}_{\overset{0}{x_1}}=\frac{\overset{\vee}{D}_{\overset{0}{x_1}}}{2}\overset{\vee}{y}_{x_1\bar{x}_1}+\frac{\overset{\vee}{D}_{\overset{0}{x_1}}}{2}\overset{\vee}{y}_{\overset{0}{x_1}}+\overset{\vee}{F}_1,\ i_1=\overline{1,N}_{1-1}\text{ for fixed j}_2, \tag{13}$$

$$y_{\overset{0}{t}}-\frac{\overset{\vee}{D}}{2}\widehat{y}_{x_2\bar{x}_2}-\frac{\overset{\vee}{D}_{\overset{0}{x_1}}}{2}\widehat{y}_{\overset{0}{x_2}}=\frac{\overset{\vee}{D}_{\overset{0}{x_2}}}{2}\overset{\vee}{y}_{x_2\bar{x}_2}+\frac{\overset{\vee}{D}_{\overset{0}{x_2}}}{2}\overset{\vee}{y}_{\overset{0}{x_2}}+\overset{\vee}{F}_2,\ i_2=\overline{1,N}_{2-1}\text{ for fixed j}_1, \tag{14}$$

$$\begin{aligned}y_{\overset{0}{t}}-\frac{\overset{\vee}{D}}{2}\left(\widehat{y}_{x_1\bar{x}_1}+\widehat{y}_{x_2\bar{x}_2}\right)-\frac{\overset{\vee}{D}_{\overset{0}{x_1}}}{2}\widehat{y}_{\overset{0}{x_1}}-\frac{\overset{\vee}{D}_{\overset{0}{x_1}}}{2}\widehat{y}_{\overset{0}{x_2}}&=\frac{\overset{\vee}{D}}{2}\left(\overset{\vee}{y}_{x_1\bar{x}_1}+\overset{\vee}{y}_{x_2\bar{x}_2}\right)+\\+\frac{\overset{\vee}{D}_{\overset{0}{x_1}}}{2}\overset{\vee}{y}_{\overset{0}{x_1}}+\frac{\overset{\vee}{D}_{\overset{0}{x_2}}}{2}\overset{\vee}{y}_{\overset{0}{x_2}}+\overset{\vee}{F}_3,\quad i_1&=\overline{1,N_1-1},\quad i_2=\overline{1,N_2-1},\end{aligned} \tag{15}$$

$$y(i_{1h_1},0,j\tau)=0,\qquad x_1\notin(\alpha,\beta);\qquad y(i,h_1,9,j\tau)=y_1,x_1\in(\alpha,\beta),$$

$$y_{\underset{x_2}{0}}\Big|_{x_2=H} = 0, \qquad y_{\underset{x_1}{0}}\Big|_{x_1=0} = 0, \qquad y_{\underset{x_1}{0}}\Big|_{x_1=L} = 0.$$

The schemes (13) and (14) are $O(\tau + h^2)$ accuracy and they one solved by the method of "progonka" which is stable if $-\frac{2D}{h} < D_{\underset{x_i}{0}} < \frac{2D}{h}, \qquad \left(\left|\frac{\partial D}{\partial x_i}\right| < \mu\right)$.

The (15) is absolutaly stable scheme and it is $O(\tau + |h|^2)$ order.

Note 1. If we denote by $y_{ij}^1,\ y_{ij}^2$ the solution which are obtained for schemes (13) and (14) then $y_{ij} = \frac{1}{2}\left(y_{ij}^1 +\ y_{ij}^2\right)$ represents solution of the difference scheme (15).

Note 2. When D represent exponent function (for instance $D = D_0 w^{\sigma}$) then preparable to use (10) then (11) or (12).

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