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ON THE CAUCHY INTEGRALS TAKEN OVER THE INFINITE LINE

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In this work the Cauchy integrals taken over the doubly-periodic line are defined and the inversion formula for this types of integrals is obtained.

In a complex z-plane C, $z = x + iy$, consider two complex numbers ω_1 and ω_2, $(\mathrm{Im}\frac{i\omega_2}{\omega_1} > 0)$ and the doubly-periodic line L which is a union of a countable number of smooth non-intersected contours L^j_{mn}; $j = 1, 2, \cdots, k$, $m, n = 0, \pm 1, \pm 2, \ldots$, doubly-periodically distributed with periods $2\omega_1$ and $2i\omega_2$ in the whole z-plane

$$\begin{aligned} L &= \bigcup_{m,n=-\infty}^{\infty} L_{mn}, \\ L_{mn} &= \bigcup_{j=1}^{k} L^j_{mn}, \quad L^{j_1}_{mn} \cap L^{j_2}_{mn} = \emptyset, \quad j_1 \neq j_2, \quad j_1, j_2 = 1, 2, \ldots, k. \end{aligned} \tag{1}$$

z-plane cut along L we denote by S.

Let $\varphi(t)$ be a doubly-periodic function given on L of Muskhelishvili H^* class on L^j_{00} [1] and consider the integral

$$\int_L \frac{\varphi(t)\,dt}{t - z}, \tag{2}$$

which is understood as the following series

$$\int_L \frac{\varphi(t)\,dt}{t - z} = \sum_{m,n=-\infty}^{\infty} \int_L \frac{\varphi(t)\,dt}{t - z}. \tag{3}$$

In the work [2] of the author the following theorem was proved.

Theorem 1. *The series* (3) *is convergent if and only if* $\varphi(t)$ *satisfies the conditions*

$$\int_{L_{00}} \varphi(t)\,dt = 0, \quad \int_{L_{00}} t\,\varphi(t)\,dt = 0,$$

and the integral (1) *is given by the formula*

$$\frac{1}{2\pi i}\int_L \frac{\varphi(t)}{t - z}\,dt = \frac{1}{2\pi i}\int_{L_{00}} \varphi(t)\,\zeta(t - z)\,dt,$$

where ζ *is the Weierstrass* ζ*-function.*

In the work the following equation is considered

$$\frac{1}{\pi i}\int_L \frac{\varphi(t)\,dt}{t - t_0} = f(t_0), \quad t_0 \in L, \tag{4}$$

with the conditions

$$\int_{L_{00}} \varphi(t)\, dt = 0, \quad \int_{L_{00}} t\, \varphi(t)\, dt = 0, \tag{5}$$

where L is the doubly-periodic line, $f(t)$ is the given doubly-periodic function of H class on every L_{00}, $m, n = 0, \pm 1, \pm 2, \ldots$), $\varphi(t)$ is unknown doubly-periodic function which is assumed to belong to the Muskhelishvili class H^*.

The solution of the problem we will find in the Muskhelishvili-Kveselava classes.

By h_q is denoted the class of solutions of the equation (4) Holder continuous on L and bounded at the ends $c_1, c_2, \ldots, c_q$ of the line L ($q \le 2k$), and at the other ends $c_{q+1}, \ldots, c_{2k}$ the solution becomes infinite with the degree less then 1 [1].

Applying the results from [2] we conclude

1) *In the case $q - k < 0$ the solution of the equation* (4) *of the class h_q exists if and only if*

$$\int_{L_{00}} f(t)\, dt = 0 \tag{6}$$

and is given by

$$\begin{aligned} \varphi(t_0) = {} & \frac{\Psi_0(t_0)}{\pi i} \int_{L_{00}} \frac{f(t)}{\Psi_0^+(t)} \left[\zeta(t - t_0) + \zeta(t_0 - \alpha_1)\right] dt + \\ & + 2\, C_1 \Psi_0^*(t_0) + 2\, C_2 \Psi_0(t_0), \quad t_0 \in L_{00}, \end{aligned} \tag{7}$$

where

$$\Psi_0(t_0) = \frac{\sigma(t_0 - \alpha_1) \cdots \sigma(t_0 - \alpha_{k-q+1})}{\sigma(t_0 - \beta_1)} \sqrt{\frac{\sigma(t_0 - c_1)\, \sigma(t_0 - c_2) \cdots \sigma(t_0 - c_q)}{\sigma(t_0 - c_{q+1}) \cdots \sigma(t_0 - c_{2k})}},$$

$$\Psi_0^*(t_0) = \frac{\sigma(t_0 - \alpha_1^*) \cdots \sigma(t_0 - \alpha_{k-q+1}^*)}{\sigma(t_0 - \alpha_1) \cdots \sigma(t_0 - \alpha_{k-q+1})}\, \Psi_0(t_0),$$

the constants $C_1, C_2, \beta_1, \alpha_1, \ldots, \alpha_{k-q+1}, \alpha_1^*, \ldots, \alpha_{k-q+1}^*$ are connected with the conditions

$$2\beta_1 + c_{q+1} + \ldots + c_{2k} = 2\alpha_1 + \ldots + 2\alpha_{k-q+1} + c_1 + \ldots + c_q,$$

$$\alpha_1 + \ldots + \alpha_{k-q+1} = \alpha_1^* + \ldots + \alpha_{k-q+1}^*,$$

$$\beta_1, \alpha_j, \alpha_j^* \notin L, \quad j = 1, 2, \ldots, k - q + 1,$$

$$\begin{aligned} & \Psi_1(C_1 \Psi^*(\beta_1) + C_2) + \\ & + \frac{\Psi_1}{2\pi i} \int_{L_{00}} \frac{f_0(t)}{\Psi_0^+(t)} \left[\zeta(t - \beta_1) - \zeta(\alpha_1 - \beta_1)\right] dt = 0, \end{aligned}$$

$$\begin{aligned} & \int_{L_{00}} \frac{t_0 \Psi_0(t_0)}{\pi i} \int_{L_{00}} \frac{f(t)}{\Psi_0^+(t)} \left[\zeta(t - t_0) + \zeta(t_0 - \alpha_1)\right] dt\, dt_0 + \\ & + 2\, C_1 \int_{L_{00}} t_0 \Psi_0^*(t_0)\, dt_0 + 2\, C_2 \int_{L_{00}} \Psi_0(t_0)\, dt_0 = 0, \end{aligned}$$

$$\Psi^*(\beta_1) = \frac{\sigma(\beta_1 - \alpha_1^*)\cdots\sigma(\beta_1 - \alpha_{k-q+1}^*)}{\sigma(\beta_1 - \alpha_1)\cdots\sigma(\beta_1 - \alpha_{k-q+1})}, \quad \Psi_1 = \mathrm{res}\Psi_0(\beta_1).$$

2) *In the case* $k - q = 0$ *the solution of the equation* (4) *of the class* h_q *exists and is given by*

$$\varphi(t_0) = \frac{\Psi_0(t_0)}{\pi i}\int_{L_{00}} \frac{f_0(t)}{\Psi_0^+(t)}\left[\zeta(t-t_0) + \zeta(t_0 - \alpha_1)\right]dt+ \tag{8}$$
$$+2\,C_1\Psi_0^*(t_0) + 2\,C_2\Psi_0(t_0), \quad t_0 \in L_{00},$$

where

$$\Psi_0(t_0) = \frac{\sigma(t_0-\alpha_1)\,\sigma(t_0-\alpha_2)}{\sigma(t_0-\beta_1)\,\sigma(t_0-\beta_2)}\sqrt{\frac{\sigma(t_0-c_1)\cdots\sigma(t_0-c_q)}{\sigma(t_0-c_{q+1})\cdots\sigma(t_0-c_{2k})}},$$

$$\Psi_0^*(t_0) = \frac{\sigma(t_0-\alpha_1^*)\,\sigma(t_0-\alpha_2^*)}{\sigma(t_0-\alpha_1)\,\sigma(t_0-\alpha_2)}\,\Psi_0(t_0),$$

$$f_0(t_0) = f(t_0) - 2A\zeta(t_0-\beta_1) + 2A\zeta(t_0-\beta_2), \quad t_0 \in L_{00},$$
$$2\beta_1 + 2\beta_2 + c_{q+1} + \ldots + c_{2k} = 2\alpha_1 + 2\alpha_2 + c_1 + \ldots + c_q,$$
$$\alpha_1 + \alpha_2 = \alpha_1^* + \alpha_2^*; \quad \beta_j, \alpha_j, \alpha_j^* \notin L,$$

$$2A\int_{L_{00}}\left[\zeta(t_0-\beta_1) - \zeta(t_0-\beta_2)\right]dt = \int_{L_{00}} f(t)\,dt,$$
$$\Psi_1(C_1\Psi^*(\beta_1) + C_2) =$$
$$= -A\Big(1 - \int_{L_{00}} \frac{[\zeta(t-\beta_1)-\zeta(\alpha_1-\beta_2)][\zeta(t-\beta_1)-\zeta(\alpha_1-\beta_1)]}{\Psi_0^+(t)}\,dt\Big) -$$
$$-\frac{\Psi_1}{2\pi i}\int_{L_{00}}\frac{f(t)}{\Psi_0^+(t)}[\zeta(t-\beta_1) - \zeta(\alpha_1-\beta_1)]\,dt,$$
$$A\Big(\frac{1}{\Psi_1} + \frac{1}{\Psi_2}\Big) = C_1(\Psi^*(\beta_2) - \Psi^*(\beta_1)),$$
$$\Psi^*(\beta_j) = \frac{\sigma(\beta_j-\alpha_1^*)\,\sigma(\beta_j-\alpha_2^*)}{\sigma(\beta_j-\alpha_1)\,\sigma(\beta_j-\alpha_2)}, \quad \Psi_j = \mathrm{res}\Psi_0(\beta_\mathrm{j}), \quad \mathrm{j} = 1, 2.$$

3) *In the case* $q - k > 0$ *the solution of the equation* (4) *of the class* h_q *exists if and only if the condition* (6) *is fulfilled and is given by*

$$\varphi(t_0) = \frac{\Psi_0(t_0)}{\pi i}\int_{L_{00}} \frac{f(t)}{\Psi_0^+(t)}\left[\zeta(t-t_0) + \zeta(t_0 - \alpha_1)\right]dt+ \tag{9}$$
$$+2C\Psi_0(t_0), \quad t_0 \in L_{00},$$

where

$$\Psi_0(z) = \frac{\sigma(z-\alpha_1)}{\sigma(z-\beta_1)\cdots\sigma(z-\beta_{q-k+1})}\sqrt{\frac{\sigma(z-c_1)\cdots\sigma(z-c_q)}{\sigma(z-c_{q+1})\cdots\sigma(z-c_{2k})}},$$

$C, \beta_{p+1}, \ldots, \beta_{q-k+1}$ *are the constants satisfying the conditions*

$$2(\beta_1 + \beta_2 + \cdots + \beta_{q-k+1}) + c_{q+1} + c_{q+2} + \cdots + c_{2k} = 2\alpha_1 + c_1 + \cdots + c_q,$$

$$\alpha_1, \beta_j \notin L, \quad j = p+1, \ldots, q-k+1,$$

$$C = -\frac{1}{2\pi i} \int_{L_{00}} \frac{f(t)}{\Psi_0^+(t)} \left[\zeta(t - \beta_1) + \zeta(\beta_1 - \alpha_1)\right] dt.$$

Note. The inversion formula of the integral equation in the case when the line L is consisted of closed contours is obtained by the author in [2].

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