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ON NUMERICAL SOLUTION OF THE BOUNDARY VALUE PROBLEM FOR UNIFORM EQUILIBRIUM EQUATIONS FOR NON-SHALLOW SPHERICAL SHELLS

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In the present paper the system of the equilibrium equations of the non-shallow (with changing internal geometry on thickness) spherical bodies of shell type are discussed, when the displacement vector is independent from the thickness coordinate [1]:

$$\begin{cases} 4\mu\dfrac{\partial}{\partial\bar{z}}\left(\dfrac{1}{\Lambda}\dfrac{\partial u_+}{\partial z}\right)+2\left(\lambda+\mu\right)\dfrac{\partial\theta}{\partial\bar{z}}+\dfrac{4\left(\lambda+2\mu\right)}{\rho}\dfrac{\partial u_3}{\partial\bar{z}}=0, \\ \mu\nabla^2u_3-\dfrac{\lambda+3\mu}{\rho}\theta-\dfrac{2\left(\lambda+2\mu\right)}{\rho^2}u_3=0, \end{cases} \tag{1}$$

$$\theta=\frac{1}{\Lambda}\left(\frac{\partial u_+}{\partial z}+\frac{\partial\overline{u_+}}{\partial\bar{z}}\right),\quad \Lambda=\frac{4\rho^2}{\left(1+z\bar{z}\right)^2}=\frac{4\rho^2}{\left(1+x^2+y^2\right)^2}$$

$$\varphi=\varphi_1+\imath\varphi_2,\quad u_+=u_1+\imath u_2,$$

where $\nabla^2=\dfrac{4}{\Lambda}\dfrac{\partial^2}{\partial z\partial\bar{z}}$ is Laplas's operator surfaces of the sphere, ρ - radius of the sphere, λ and μ are the elasticity constants lame, $u_1,\ u_2,\ u_3$ components of the vector of the displacement u, $z=x+\imath y$ complex variable, $\varphi_1,\ \varphi_2,\ \varphi_3$ components of external force.

From the system (1) to find into account the following equalities:

$$\frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x}-\imath\frac{\partial}{\partial y}\right),\quad \frac{\partial}{\partial\bar{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x}+\imath\frac{\partial}{\partial y}\right),$$

$$\theta=\frac{1}{\Lambda}\left(\frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial y}\right),\quad \nabla^2=\frac{1}{\Lambda}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right),$$

we can rewrite the system of the equilibrium equations (1) as

$$\begin{cases} a_1u_{1,xx}+u_{1,yy}+f_1(x,y)(u_{1,x}+u_{2,y})-f_2(x,y)(u_{2,x}-u_{1,y}) \\ \qquad\qquad +a_2u_{2,xy}+f_3(x,y)u_{3,x}=0, \\ u_{2,xx}+a_1u_{2,yy}+f_4(x,y)(u_{2,x}-u_{1,y})+f_5(x,y)(u_{1,x}+u_{2,y}) \\ \qquad\qquad +a_2u_{1,xy}+f_3(x,y)u_{3,y}=0, \\ \mu(u_{3,xx}+u_{3,yy})-a_3(u_{1,x}+u_{2,y})-\dfrac{\mu}{\rho}f_3(x,y)u_3=0, \end{cases} \tag{2}$$

where

$$a_1 = \frac{\lambda + 2\mu}{\mu}, \quad a_2 = \frac{\lambda + \mu}{\mu}, \quad a_3 = \frac{\lambda + 3\mu}{\mu},$$

$$f_1(x,y) = \frac{4a_1 x}{1 + x^2 + y^2}, \quad f_2(x,y) = \frac{4y}{1 + x^2 + y^2}, \quad f_3(x,y) = \frac{8\rho a_1}{(1 + x^2 + y^2)^2},$$

$$f_4(x,y) = \frac{4x}{1 + x^2 + y^2}, \quad f_5(x,y) = \frac{4a_1 y}{1 + x^2 + y^2}, \quad f_6(x,y) = \frac{4\rho^2}{(1 + x^2 + y^2)^2}.$$

Let us assume that in the rectangle $G = \{0 < x < 1,\ 0 < y < 1\}$ with boundary Γ it is necessary to find the solutions of the equation (2), satisfying conditions

$$u_1(x,y)|_\Gamma = 1, \quad u_2(x,y)|_\Gamma = 2, \quad u_3(x,y)|_\Gamma = 3. \tag{3}$$

We shall define the square grid

$$G_h = \left\{ x_{ij} = \left(x^{(i)}, y^{(j)} \right) \right\},$$

where $x^{(i)} = ih, \quad y^{(j)} = jh, \quad i = 0, 1, ..., N, \quad j = 0, 1, ..., N, \quad hN = 1.$

We shall change the problem (2)-(3) by differential scheme [2]

$$\begin{cases} a_1 \dfrac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2} + \dfrac{u_i^{j+1} - 2u_i^j + u_i^{j-1}}{h^2} + f_{1,i}^j \left(\dfrac{u_{i+1}^j - u_i^j}{h} + \dfrac{v_i^{j+1} - v_i^j}{h} \right) \\ -f_{2,i}^j \left(\dfrac{v_{i+1}^j - v_i^j}{h} - \dfrac{u_i^{j+1} - u_i^j}{h} \right) + a_2 \dfrac{v_{i+1}^{j+1} - v_{i+1}^j - v_i^{j+1} + v_i^j}{h^2} + f_{3,i}^j \dfrac{w_{i+1}^j - w_i^j}{h} = 0, \\ a_1 \dfrac{v_i^{j+1} - 2v_i^j + v_i^{j-1}}{h^2} + \dfrac{v_{i+1}^j - 2v_i^j + v_{i-1}^j}{h^2} + f_{4,i}^j \left(\dfrac{v_{i+1}^j - v_i^j}{h} + \dfrac{u_i^{j+1} - u_i^j}{h} \right) \\ +f_{5,i}^j \left(\dfrac{u_{i+1}^j - u_i^j}{h} + \dfrac{v_i^{j+1} - v_i^j}{h} \right) + a_2 \dfrac{u_{i+1}^{j+1} - u_{i+1}^j - u_i^{j+1} + u_i^j}{h^2} + f_{3,i}^j \dfrac{w_i^{j+1} - w_i^j}{h} = 0, \\ \mu \left(\dfrac{w_{i+1}^j - 2w_i^j + w_{i-1}^j}{h^2} + \dfrac{w_i^{j+1} - 2w_i^j + w_i^{j-1}}{h^2} \right) - a_3 \left(\dfrac{u_{i+1}^j - u_i^j}{h} + \dfrac{v_i^{j+1} - v_i^j}{h} \right) \\ \qquad -\dfrac{\mu}{\rho} f_{3,i}^j w_i^j = 0, \end{cases} \tag{4}$$

$$i = 1, 2, ..., N-1, \quad j = 1, 2, ..., N-1,$$

where boundary conditions have the following forms:

$$\begin{gathered} u_{oj} = 1, \quad u_{Nj} = 1, \quad u_{io} = 1, \quad u_{iN} = 1, \\ v_{oj} = 2, \quad v_{Nj} = 2, \quad v_{io} = 2, \quad v_{iN} = 2, \\ w_{oj} = 3, \quad w_{Nj} = 3, \quad w_{io} = 3, \quad w_{iN} = 3, \\ i = 1, ..., N-1, \quad j = 1, ..., N-1. \end{gathered} \tag{5}$$

Using the notations

$$\Lambda_0 = -\left(\frac{2a_1}{h^2} + \frac{f_{1,i}^j}{h}\right)E, \quad \Lambda_1 = \frac{1}{h^2}T, \quad \Lambda_2 = \frac{f_{2,i}^j}{h}\Pi, \quad \Lambda_3 = \left(\frac{a_1}{h^2} + \frac{f_{1,i}^j}{h}\right)E,$$

$$\Lambda_4 = \left(\frac{f_{1,i}^j}{h} - \frac{a_2}{h^2}\right)\Pi, \quad \Lambda_5 = \frac{f_{2,i}^j}{h}E, \quad \Lambda_6 = \frac{a_2}{h^2}\Pi, \quad \Lambda_7 = \frac{f_{3,i}^j}{h}E,$$

$$K_0 = \left(\frac{f_{4,i}^j}{h} + \frac{a_2}{h^2}\right)\Pi, \quad K_1 = \frac{f_{5,i}^j}{h}E, \quad K_2 = \frac{a_2}{h^2}\Pi, \quad K_3 = -\left(\frac{f_{4,i}^j}{h} + \frac{2}{h^2}\right)E,$$

$$K_4 = \frac{a_1}{h^2}T, \quad K_5 = \frac{f_{5,i}^j}{h}\Pi, \quad K_6 = \left(\frac{1}{h^2} + \frac{f_{4,i}^j}{h}\right)E, \quad K_7 = \frac{f_{3,i}^j}{h}\Pi,$$

$$M_0 = \frac{a_3}{h}\Pi, \quad M_1 = \left(\frac{2\mu}{h^2} + \frac{4\mu}{\rho}f_{3,i}^j\right)E, \quad M_1 = \frac{\mu}{h^2}T,$$

where

$$\Pi = \begin{pmatrix} -1 & 1 & 0 & 0...0 \\ 0 & -1 & 0 & 0...0 \\ & & ... & \\ 0 & 0 & 0 & 0...-1 \end{pmatrix}, \quad T = \begin{pmatrix} -2 & 1 & 0 & 0...0 & 0 \\ 1 & -2 & 1 & 0...0 & 0 \\ & & ... & & \\ 0 & 0 & 0 & 0...1 & -2 \end{pmatrix}$$

we can rewrite (4) system as [3]

$$\begin{cases} \frac{a_1}{h^2}Eu_{i-1} + [\Lambda_0 + \Lambda_1 + \Lambda_2]\,u_i + \Lambda_3 u_{i+1} + [\Lambda_4 + \Lambda_5]\,v_i \\ \qquad + [-\Lambda_5 + \Lambda_6]\,v_{i+1} - \Lambda_7\,[w_i - w_{i+1}] = 0, \\ -[K_0 + K_1]\,u_i + [K_1 + K_2]\,u_{i+1} + \frac{1}{h^2}Ev_{i-1} \\ \qquad + [K_3 + K_4 + K_5]\,v_i + K_6 v_{i+1} + K_7 w_i = 0, \\ \frac{a_3}{h}E\,(u_i - u_{i+1}) - M_0 v_i + \frac{\mu}{h^2}Ew_{i-1} + [-M_1 + M_2]\,w_i + \frac{\mu}{h^2}Ew_{i+1} = 0, \end{cases} \tag{6}$$

where

$$u = (u_1, ..., u_{N-1})^T, \quad v = (v_1, ..., v_{N-1})^T, \quad w = (w_1, ..., w_{N-1})^T.$$

From the system (6) we have:

$$A_{11}u_{i-1} + A_{12}u_i + A_{13}u_{i+1} + B_{11}v_{i-1} + B_{12}v_i + B_{13}v_{i+1}$$
$$+C_{11}w_{i-1} + C_{12}w_i + C_{13}w_{i+1} = 0,$$
$$A_{21}u_{i-1} + A_{22}u_i + A_{23}u_{i+1} + B_{21}v_{i-1} + B_{22}v_i + B_{23}v_{i+1}$$
$$+C_{21}w_{i-1} + C_{22}w_i + C_{23}w_{i+1} = 0,$$

$$A_{31}u_{i-1} + A_{32}u_i + A_{33}u_{i+1} + B_{31}v_{i-1} + B_{32}v_i + B_{33}v_{i+1}$$
$$+C_{31}w_{i-1} + C_{32}w_i + C_{33}w_{i+1} = 0,$$
$$i = 1, ..., N-1,$$

where

$$A_{11} = \frac{a_1}{h^2}E,\ A_{12} = \Lambda_0 + \Lambda_1 + \Lambda_2,\ A_{13} = \Lambda_3,\ A_{21} = 0,\ A_{22} = -K_0 - K_1,$$
$$A_{23} = K_1 + K_2,\ A_{31} = 0,\ A_{32} = \frac{a_3}{h}E,\ A_{33} = -\frac{a_3}{h}E,$$
$$B_{11} = 0,\ B_{12} = \Lambda_4 + \Lambda_5,\ B_{13} = -\Lambda_5 + \Lambda_6,\ B_{21} = \frac{1}{h^2}E,$$
$$B_{22} = K_3 + K_4 + K_5,\ B_{23} = K_6,\ B_{31} = 0,\ B_{32} = -M_0,\ B_{33} = 0,$$
$$C_{11} = 0,\ C_{12} = -\Lambda_7,\ C_{13} = \Lambda_7,\ C_{21} = 0,\ C_{22} = K_7,$$
$$C_{23} = 0,\ C_{31} = \frac{\mu}{h^2}E,\ C_{32} = -M_1 + M_2,\ C_{33} = \frac{\mu}{h^2}E.$$

Let us [4]

$$\rho = 1,\quad \sigma = 0,3,\quad E = 2,1 \times 10^6,$$
$$\mu = \frac{E}{2(1+\sigma)} = 8,077 \times 10^5,\quad \lambda = \frac{\sigma}{1-\sigma^2} = 0,33.$$

Then the approximate solutions of the system (2)-(3) have the following forms:

$u_1 = -15.503$	$v_1 = 8.808$	$w_1 = 2.947$
$u_3 = -47.2$	$v_3 = 23.08$	$w_3 = 2.902$
$u_6 = -14.173$	$v_6 = 9.309$	$w_6 = 2.917$
$u_{10} = -92$	$v_{10} = 60.834$	$w_{10} = 2.841$
$u_{15} = -48.775$	$v_{15} = 81.99$	$w_{15} = 2.902$
$u_{17} = -122.874$	$v_{17} = 131.466$	$w_{17} = 2.808$
$u_{20} = -34.791$	$v_{20} = 16.753$	$w_{20} = 2.841$
$u_{25} = -89.308$	$v_{25} = 122.869$	$w_{25} = 2.786$
$u_{29} = -121.725$	$v_{29} = 278.659$	$w_{29} = 2.902$
$u_{34} = -24.382$	$v_{34} = 34.507$	$w_{34} = 2.841$
$u_{37} = -64.354$	$v_{37} = 212.685$	$w_{37} = 2.868$
$u_{40} = 24.44$	$v_{40} = 56.222$	$w_{40} = 2.841$
$u_{44} = -30.678$	$v_{44} = 65.517$	$w_{44} = 2.917$
$u_{49} = -0.938$	$v_{49} = 7.462$	$w_{49} = 2.947$

Therefore, the boundary value problem with the help of the method of finite-difference, for the rectangular area, for non-shallow spherical shells has been solved.

R E F E R E N C E S

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