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ON THE MODELING OF THE EUROPEAN OPTION PRICING THEORY

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1. We consider the (B, S)-financial market consisting of two kinds of assets: the bank account (bonds) $B = (B_n)$ and the stocks $S = (S_n)$, $n = 0, 1, \dots, N$. According to the well known Cox–Ross–Rubinstein binomial model, the behavior of these variables with respect to time can be expressed in terms of the recurrent relations

$$B_n = (1 + r)B_{n-1}, \quad S_n = (1 + \rho_n)S_{n-1}, \quad B_0 > 0, \quad S_0 > 0, \tag{1}$$

where $r > 0$ is an interest rate and $\rho = (\rho_n)$ is a sequence of independent, identically distributed random variables taking only two values a and b, $-1 < a < r < b$, $n = 0, 1 \dots, N$.

The European type standard option with the payoff function

$$f_N = f(S_N) = \max(S_N - K, 0) \tag{2}$$

is a bank-eligible security which can be used to buy a stock at a fixed time moment N and at an a priori prescribed price $K > 0$.

Let us assume that the investors initial capital is $X_0 = x > 0$ and we have a sequence of positive functions $g = (g_n)$, $n = 0, 1, \dots, N$, $g_0 = 0$.

Assume that at a time moment $n - 1$ the investor constructed the strategy $\pi_{n-1} = (\beta_{n-1}, \gamma_{n-1})$ (portfolio), where β_{n-1} and γ_{n-1} are respectively the number of bonds and the number of stocks. If at a time moment $n - 1$ the process of bonds and stocks are respectively B_{n-1} and S_{n-1}, then the investors capital has the form

$$X^{\pi}_{n-1} = \beta_{n-1}B_{n-1} + \gamma_{n-1}S_{n-1}. \tag{3}$$

We will construct a new minimal strategy $\pi^*_n = (\beta^*_n, \gamma^*_n)$ such that the equalities

$$X^{\pi^*}_{n-1} = \beta^*_n B_{n-1} + \gamma^*_n S_{n-1} + g_n, \tag{4}$$

$$X^{\pi^*}_N = \beta^*_N B_N + \gamma^*_N S_N = f(S_N) \tag{5}$$

be fulfilled. As to the option price, it is an initial sum such that guarantees the fulfillment of equality (5). The option price is denoted by C_N.

2. Suppose we consider the standard option with the payoff function (2) and

$$g_n = c_1\beta_n B_{n-1} + c_2\gamma_n S_{n-1}, \quad 0 < c_1 < 1, \quad 0 < c_2 < 1. \tag{6}$$

Lemma 1. *At each time moment n, $n = 0, 1, \dots, N - 1$, a minimal strategy $\pi^*_{n+1} = (\beta^*_{n+1}, \gamma^*_{n+1})$ is defined by the equalities*

$$\beta^*_{n+1} = \frac{(1 + b)f((1 + a)S_n) - (1 + a)f((1 + b)S_n)}{(1 + r)(b - a)B_n}, \tag{7}$$

$$\gamma_{n+1}^* = \frac{f((1+a)S_n) - f((1+b)S_n)}{(b-a)S_n}\,. \tag{8}$$

Proof. Assume that at some moment of time n we have the portfolio $\pi_n = (\beta_n, \gamma_n)$. We need to construct a portfolio $\pi_{n+1} = (\beta_{n+1}, \gamma_{n+1})$ such that

$$X_{n+1}^{\pi} = \beta_{n+1}B_{n+1} + \gamma_{n+1}S_{n+1} = f(S_{n+1})$$

be fulfilled at a moment of time $n+1$.

Then, taking into the financial (B, S)-market model (1), for the unknown values β_{n+1} and γ_{n+1} we obtain a system of two unknown linear equations, the solution of which $(\beta_{n+1}^*, \beta_{n+1}^*)$ is given by equalities (7) and (8).

Lemma 2. *The capital of the minimal strategy constructed by equalities* (7), (8) *is defined by the equality*

$$X_n^{\pi^*} = \frac{1+c_1}{1+r}\,[p^* f((1+b)S_n) + (1-p^*)f((1+a)S_n)], \tag{9}$$

where

$$p^* = \frac{r - c_1(1+a) + c_2(1+r) - a}{(b-a)(1+c_1)}. \tag{10}$$

Proof. Assume that the portfolio $\pi_{n+1} = (\beta_{n+1}, \gamma_{n+1})$ is constructed at a moment of time n. Then its corresponding capital can be written in the form

$$X_n^{\pi} = \beta_{n+1}B_n + \gamma_{n+1}S_n.$$

If in this expression the values $n+1$ and $n+1$ are replaced by the values β_{n+1}^* and γ_{n+1}^* defined by equalities (7) and (8), then we easily obtain the capital process expression (9) which actually represents the corresponding amount of the portfolio $\pi_{n+1}^* = (\beta_{n+1}^*, \gamma_{n+1}^*)$ at the moment of time n.

Lemma 3. *The following recurrent equalities are valid:*

$$C_{N-k,j} = \frac{1+c_1}{1+r}\,[p^* C_{N-k+1,j+1} + (1-p^*)C_{N-k+1,j}], \tag{11}$$

where $k = 1, \ldots, N$, $j = 0, 1, \ldots, N-k$, the value p^* is defined by equality (10) *and* $C_{0,0} = C_N$.

Proof. We use the method of construction of a binomial tree with N steps and the terminal node $N+1$. At a moment of time $n = 1, 2, \ldots, N$ the stock cost can be calculated by the equalities $S_N = S_{N,j} = S_0(1+b)^j(1+a)^{N-j}$, $j = 0, 1, \ldots, N$.

At the terminal moment of time $n = N$, the option prices at a node $N+1$ are calculated by the equalities $f_N = f_{N,j} = f(S_{N,j})$, $j = 0, 1, \ldots, N$.

Now, at a moment of time $N-1$, the current option prices at the preceding node N are calculated by the equalities

$$C_{N-1,j} = \frac{1+c_1}{1+r}\,[p^* f_{N,j+1} + (1-p^*)f_{N,j}], \quad j = 0, 1, \ldots, N-1.$$

Analogously, at a time moment $N-2$ we will have at an $N-1$ node

$$C_{N-2,j} = \frac{1+c_1}{1+r}\,[p^* C_{N-1,j+1} + (1-p^*)C_{N-1,j}], \quad j = 0, 1, \dots, N-2.$$

Continuing this procedure, we easily obtain relation (11) for a node $N-k+1$. If we continue the procedure in this manner, then at a moment of time $n = N - N = 0$ we reach the initial node or the vertex of the binomial tree with N steps, where the true price of the option is calculated by the equality

$$C_N =_{0,0}= \frac{1+c_1}{1+r}\,[p^* C_{1,1} + (1-p^*)C_{1,0}].$$

Theorem. *An option price is defined by the equalities*

$$C_N = S_0 \sum_{k=k_0}^{N} C_N^k (p^*)^k (1-p^*)^{N-k} \Big(\frac{(1+c_1)(1+a)}{1+r}\Big)^N \Big(\frac{1+b}{1+a}\Big)^k - \tag{12}$$

$$-K\Big(\frac{1+c_1}{1+r}\Big)^N \sum_{k=k_0}^{N} C_N^k (p^*)^k (1-p^*)^{N-k},$$

where k_0 is the smallest integer number for which there holds the inequality

$$S_0(1+a)^N \Big(\frac{1+b}{1+a}\Big)^{k_0} > K.$$

Proof. Assume that f is some payoff function and p is a number such that $0 < p < 0$. Let us introduce the notation

$$F_n(x;p) = \sum_{k=0}^{n} f\Big(x(1+b)^k(1+a)^{n-k} C_n^k p^k (1-p)^{n-k}\Big).$$

In that case, if f is a payoff function of a European type standard put option, then we have

$$F_N(S_0;p^*) =$$

$$= \sum_{k=0}^{N} C_N^k (p^*)^k (1-p^*)^{N-k} \max\Big(0, S_0(1+a)^N \Big(\frac{1+b}{1+a}\Big)^k (1+c_1)^N - K\Big).$$

If $k_0 > N$, then it can be easily shown that $F_N(S_0;p^*) = 0$, while if $k_0 \le N$, then relation (12) is fulfilled.

The lemmas are proved by means of the so-called binomial trees and the reciprocal portfolio principle, while the theorem is proved by using [1] and [2].

3. Let us now consider the binomial trees and, using the obtained formulas, solve the one-step $N = 1$, $n = 0, 1$ and two-step $N = 2$, $n = 0, 1, 2$ problems. We introduce the notation

$$S_1 = S_{1,j} = S_0(1+b)^j(1+a)^{1-j}, \quad f_1 = f_{1,j} = f(S_{1,j}), \quad j = 0, 1, \tag{13}$$

$$S_2 = S_{2,j} = S_0(1+b)^j(1+a)^{2-j}, \quad f_2 = f_{2,j} = f(S_{2,j}), \quad j = 0, 1, 2. \tag{14}$$

It is assumed that $B_0 = 20$, $r = \frac{1}{5}$, $K = 100$, $S_0 = 100$, $\rho_n = b = \frac{3}{5}$, or $\rho_n = a = -\frac{2}{5}$, $n = 0, 1, 2$.

Example 1. $N = 1$, $n = 0, 1$; $c_1 = \frac{1}{40}$, $c_2 = \frac{1}{50}$. We have $C_2 = \frac{609}{20}$, $\beta_1^* = -\frac{3}{2}$, $\gamma_1^* = \frac{3}{5}$, $g_1 = \frac{9}{20}$, $X_0^{\pi^*} = C_1$.

1) if $S_1 = S_{1,1} = 160$, then $X_1^{\pi^*} = f(S_1) = 60$;

2) if $S_1 = S_{1,0} = 60$, then $X_1^{\pi^*} = f(S_1) = 0$.

Example 2. $N = 2$, $n = 0, 1, 2$; $c_1 = \frac{1}{40}$, $c_2 = \frac{1}{50}$. We have

$$C_2 = \frac{609 \cdot 203 \cdot 13}{40000}, \quad \beta_1^* = -\frac{609 \cdot 13}{4000}, \quad \gamma_1^* = \frac{609 \cdot 13}{10000},$$
$$g_1 = \frac{609 \cdot 13 \cdot 3}{40000}, \quad X_0^{\pi^*} = C_2.$$

Case I. $S_1 = S_{1,1} = 160$, $\beta_2^* = -\frac{13}{4}$, $\gamma_2^* = \frac{39}{40}$, $g_2 = \frac{117}{100}$.

1) if $S_2 = S_{2,2} = 256$, then $X_2^{\pi^*} = f(S_2) = 156$;

2) if $S_2 = S_{2,1} = 96$, then $X_2^{\pi^*} = f(S_2) = 0$.

Case II. $S_1 = S_{1,0} = 60$, $\beta_2^* = \gamma_2^* = g_2 = 0$.

1) if $S_2 = S_{2,1} = 96$, then $X_2^{\pi^*} = f(S_2) = 0$;

2) if $S_2 = S_{2,0} = 36$, then $X_2^{\pi^*} = f(S_2) = 0$.

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