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TWO ELASTIC SOLIDS INTERACTION PROBLEM

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Abstract

The contact problem for homogeneous equation systems of classic and asymmetrical theory is considered, when a spare is the contact surface. it is proved, that the problem has a singular solution. The solution is presented in the form of absolutely and uniformly convergent series.

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Let Ω^+ be the spere bounded by spherical surface $\partial\Omega$ with the center in the origin and the radius R.

Problem (C). Find the regular vectors $u^{(0)}(x)$ and $u^{(1)}(x)$ in Ω^+ and Ω^- areas accordingly, satisfying:

1. equations [1.3]

$$M(\partial x)u^{(o)}(x) = \Delta(\mu_0 - \nu_0\Delta)u^{(0)}(x) + (\lambda_0 + \mu_0 + \nu_0\Delta)\operatorname{grad}\operatorname{div} u^{(0)}(x) = 0, \quad x \in \Omega^+,$$

$$A(\partial x)u^{(1)}(x) = \mu_1\Delta u^{(1)}(x) + (\lambda_1 + \mu_1)\operatorname{grad}\operatorname{div} u^{(1)}(x) = 0, \quad x \in \Omega^-, \tag{1}$$

2. boundary conditions

$$\begin{aligned} &[u^{(0)}(z)]^+ - [u^{(1)}(z)]^- = f^{(1)}(z), \quad z \in \partial\Omega, \\ &[T^{(0)}(\partial z, n)u^{(0)}(z)]^+ - [T^{(1)}(\partial z, n)u^{(1)}(z)]^- = f^{(2)}(z), \quad z \in \partial\Omega, \\ &[R^{(0)}(\partial z, n)u^{(0)}(z)]^+ = f^{(3)}(z), \quad z \in \partial\Omega, \end{aligned} \tag{2}$$

3. Vector $u^{(1)}(x)$ satisfies nearby infinitely distant pint the conditions:

$$u_j^{(1)}(x) = O(|x|^{-1}), \quad \frac{\partial u_j^{(1)}(x)}{\partial x_k} = o(|x|^{-1}), \quad k, j = 1, 2, 3, \tag{3}$$

where $f^{(j)} = (f_1^{(j)},\ f_2^{(j)},\ f_3^{(j)},)$, $\quad j = 1, 2, 3$ are the given vectors on $\partial\Omega$, $n(z)$ is the ort of normal outside with regard to Ω^+ in point $z \in \partial\Omega$. $\quad u^{(j)}(x) = (u_1^{(j)}(x),\ u_2^{(j)}(x),\ u_3^{(j)}(x))$, $j = 0, 1$- the vector if displacement, Δ-laplace operator.

$$T^{(0)}(\partial x, n)u^{(0)} = 2\mu_0\frac{\partial u^{(0)}}{\partial n} + \lambda_0 n\operatorname{div} u^{(0)} + \mu_0[n \times \operatorname{rot} u^{(0)}] + \nu_0[n \times \Delta\operatorname{rot} u^{(0)}]$$
$$-\frac{1}{2}[n \times \operatorname{grad}(n \cdot q(u^{(0)}))],$$

$$
\begin{aligned}
&T^{(0)}(\partial x, n)u^{(1)} = 2\mu_1 \frac{\partial u^{(1)}}{\partial n} + \lambda_1 n \operatorname{div} u^{(1)} + \mu_1 [n \times \operatorname{rot} u^{(1)}],\\
&R(\partial x, n)u^{(0)} = \operatorname{rot} u^{(0)} - n(n \cdot \operatorname{rot} u^{(0)}),\\
&q(u^{(0)}) = 2(\nu_0 + \nu_0') \frac{\partial}{\partial n} \operatorname{rot} u^{(0)} + 2\nu' [n \times \operatorname{rot}\operatorname{rot} u^{(0)}],
\end{aligned}
$$

$\lambda_j,\ \mu_j,\ \nu_0,\ \nu_0',\quad j = 0, 1$-elastic constants, satisfying the following conditions:

$$
\mu_j > 0, \quad 3\lambda_j + 2\mu_j > 0, \quad |\nu_0'| \leq \nu_0, \quad j = 0, 1.
$$

Theorem 1. *The problem (C) permits not more than one regular solution.*

Proof. The theorem will be proved, if we show that the corresponding homogeneous problem ($f^{(j)}(z) = 0, \quad j = 1, 2, 3$) has only trivial solution. Green's formula in Ω^- domain area for system (1) has a form [3,5]

$$
\begin{aligned}
\int\limits_{\Omega^-} u^{(1)}(x) \cdot A(\partial x) u^{(1)}(x)\, dx = -\int\limits_{\partial\Omega} [u^{(1)}(z)]^- \cdot [T^{(1)}(\partial z, n) u^{(1)}(z)]^- ds \\
- \int\limits_{\Omega^-} \widetilde{E}^{(1)}(u^{(1)}, u^{(1)})\, dx,
\end{aligned}
\tag{4}
$$

keeping in mind boundary conditions of problem $(C)_0$,we'll have:

$$
\begin{aligned}
\int\limits_{\Omega^+} u^{(0)}(x) \cdot M(\partial x) u^{(0)}(x)\, dx = \int\limits_{\partial\Omega} \{[u^{(0)}(z)]^+ \cdot [T^{(0)}(\partial z, n) u^{(0)}(z)]^+ \\
+ [R(\partial z, n) u^{(0)}(z)]^+ \cdot [Q^{(0)}(\partial z, n) u^{(0)}(z)]^+\}\, ds - \int\limits_{\Omega^+} E^{(0)}(u^{(0)}, u^{(0)})\, dx,
\end{aligned}
\tag{5}
$$

where [3,5]

$$
\begin{aligned}
&\widetilde{E}^{(1)}(u^{(1)}, u^{(1)}) \geq 0, \quad E^{(0)}(u^{(0)}, u^{(0)}) \geq 0,\\
&\omega_k^{(0)} = \frac{1}{2} (\operatorname{rot} u^{(0)})_k, \quad k = 1, 2, 3,\\
&Q^{(0)}(\partial x, n) u^{(0)} = \frac{1}{2} \Big[q(u^{(0)}) - n(n \cdot q(u^{(0)})) \Big].
\end{aligned}
$$

Keeping in mind boundary conditions of problem $(C)_0$, we'll have:

$$
E^{(0)}(u^{(0)}, u^{(0)}) = 0, \quad \widetilde{E}^{(1)}(u^{(1)}, u^{(1)}) = 0.
$$

The solutions of these equations have the form [3,5]:

$$
\begin{aligned}
&u^{(0)}(x) = [a^{(0)} \times x] + b^{(0)}, \quad x \in \Omega^+,\\
&u^{(1)}(x) = [a^{(1)} \times x] + b^{(1)}, \quad x \in \Omega^-,
\end{aligned}
$$

Taking into consideration conditions (3) and boundary conditions of problem $(C)_0$, we'll have: $u^{(0)}(x) = 0, \quad x \in \Omega^+, \quad u^{(0)}(x) = 0, \quad x \in \Omega^-.\Box$

The solution of problem (C) we'll search in the form [5]:

$$\begin{aligned}
u^{(0)}(x) &= \operatorname{grad}\Phi_1^{(0)}(x) - a_0 \operatorname{grad} r^2\Big(r\frac{\partial}{\partial r}+1\Big)\Phi_2^{(0)}(x) + \operatorname{rot}\operatorname{rot}(xr^2\Phi_2^{(0)}(x)) \\
&\quad + \operatorname{rot}(x\Phi_3^{(0)}(x)) + \operatorname{rot}\operatorname{rot}(x\Phi_4^{(0)}(x)) + \operatorname{rot}(x\Phi_5^{(0)}(x)), \quad x \in \Omega^+, \\
u^{(1)}(x) &= \operatorname{grad}\Phi_1^{(1)}(x) - a_1 \operatorname{grad} r^2\Big(r\frac{\partial}{\partial r}+1\Big)\Phi_2^{(1)}(x) + \operatorname{rot}\operatorname{rot}(xr^2\Phi_2^{(1)}(x)) \\
&\quad + \operatorname{rot}(x\Phi_3^{(1)}(x)) \quad x \in \Omega^+,
\end{aligned} \tag{6}$$

where $\Phi_j^{(l)}(x), \quad l = 1,2, \quad j = 1,2,3$-scalar harmonic functions, $\Phi_j^{(0)}(x), \quad j = 4,5$-scalar metaharmonic functions.

$$\begin{gathered}
(\Delta - l^2)\Phi_j^{(0)}(x) = 0, \ \ j = 4,5, \ \ l^2 = \frac{\mu}{\nu}, a_l = \mu(\lambda_l + 2\mu_l)^{-1}, \ \ l = 0,1, \\
x = (x_1, x_2, x_3), \ \ r = |x|, \ \ r\frac{\partial}{\partial r} = x \cdot \operatorname{grad}.
\end{gathered}$$

We write down functions $\Phi_j^{(l)}(x)$ in the form:

$$\begin{aligned}
\Phi_j^{(0)}(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big(\frac{r}{R}\Big)^k Y_k^{(m)}(\theta,\varphi)A_{mk}^{(j)}, \quad j = 1,2,3, \\
\Phi_j^{(0)}(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k} g_k(lr) Y_k^{(m)}(\theta,\varphi)A_{mk}^{(j)}, \quad j = 4,5, \\
\Phi_j^{(1)}(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big(\frac{R}{r}\Big)^k Y_k^{(m)}(\theta,\varphi)B_{mk}^{(j)}, \quad j = 1,2,3,
\end{aligned} \tag{7}$$

where (r,θ,φ)-spherical coordinates of the point x; $A_{mk}^{(j)}, B_{mk}^{(j)}$-desired constants

$$\begin{aligned}
Y_k^{(m)}(\theta,\varphi) &= \sqrt{\frac{2k+1}{4\pi}\cdot\frac{(k-m)!}{(k+m)!}}P_k^{(m)}(\cos\theta)e^{im\varphi}, \\
g_k(lr) &= \sqrt{\frac{R}{r}}\frac{I_{k+\frac{1}{2}}(lr)}{I_{k+\frac{1}{2}}(lR)},
\end{aligned}$$

$P_k^{(m)}(\cos\theta)$-Lejandr's added function of the first type, $I_{k+\frac{1}{2}}(lr)$-Besel's function of imaginary argument.

if the value of functions $\Phi_j^{(l)}(x)$ put from (7) to (6), we'll have:

$$u^{(j)}(x) = \sum_{k=o}^{\infty}\sum_{m=-k}^{k}\{u_{mk}^{(j)}(r)X_{mk}(\theta,\varphi) + \sqrt{k(k+1)}[v_{mk}^{(j)}(r)Y_{mk}(\theta,\varphi)$$

$$+w_{mk}^{(j)}(r)Z_{mk}(\theta,\varphi)]\},$$

$$T^{(j)}(\partial x,n)u^{(j)}(x)=\sum_{k=o}^{\infty}\sum_{m=-k}^{k}\{a_{mk}^{(j)}(r)X_{mk}(\theta,\varphi)+\sqrt{k(k+1)}[b_{mk}^{(j)}(r)Y_{mk}(\theta,\varphi) \tag{8}$$

$$+c_{mk}^{(j)}(r)Z_{mk}(\theta,\varphi)]\},\quad j=0,1,$$

$$R(\partial x,n)u^{(0)}(x)=\sum_{k=1}^{\infty}\sum_{m=-k}^{k}\sqrt{k(k+1)}[b_{mk}^{(2)}(r)Y_{mk}(\theta,\varphi)+c_{mk}^{(2)}(r)Z_{mk}(\theta,\varphi)]$$

Where $u_{mk}^{(j)}(r),\ldots,c_{mk}^{(j)}(r)$, $b_{mk}^{(2)}(r)$, $c_{mk}^{(2)}(r)$, $j=0,1$-functions of r, $X_{mk}(\theta,\varphi)$, $Y_{mk}(\theta,\varphi)$, $Z_{mk}(\theta,\varphi)$-ortonormalized vectors in $L_2(\Sigma_1)$ class [4].

Let vector $f^{(j)}(z)$, $j=1,2,3$ satisfy the conditions, under which we can spread it out by Fourie-Laplass series in system

$$\{X_{mk}(\theta,\varphi),\ Y_{mk}(\theta,\varphi),\ Z_{mk}(\theta,\varphi)\}_{|m|\le k,\ k=\overline{0,\infty}},$$

$$f^{(j)}(z)=\sum_{k=o}^{\infty}\sum_{m=-k}^{k}\{\alpha_{mk}^{(j)}X_{mk}(\theta,\varphi)+\sqrt{k(k+1)}[\beta_{mk}^{(j)}Y_{mk}(\theta,\varphi)+\gamma_{mk}^{(j)}Z_{mk}(\theta,\varphi)]\}, \tag{9}$$

$$f^{(3)}(z)=\sum_{k=1}^{\infty}\sum_{m=-k}^{k}\sqrt{k(k+1)}[\beta_{mk}^{(3)}Y_{mk}(\theta,\varphi)+\gamma_{mk}^{(3)}Z_{mk}(\theta,\varphi)].\ \ j=1,2,$$

proceeding to limit in both sides of the equality (8), when $x\to z\in\partial\Omega$ keeping in mind boundary conditions (2), formulas (9) also, we'll have following system for desired constants:

$$\begin{array}{ll}
u_{mk}^{(0)}(R)-u_{mk}^{(1)}(R)=\alpha_{mk}^{(1)}, & a_{mk}^{(0)}(R)-a_{mk}^{(1)}(R)=\alpha_{mk}^{(2)},\ \ k\ge 0,\\
v_{mk}^{(0)}(R)-v_{mk}^{(1)}(R)=\beta_{mk}^{(1)}, & b_{mk}^{(0)}(R)-b_{mk}^{(1)}(R)=\beta_{mk}^{(2)},\\
w_{mk}^{(0)}(R)-w_{mk}^{(1)}(R)=\gamma_{mk}^{(1)}, & c_{mk}^{(0)}(R)-c_{mk}^{(1)}(R)=\gamma_{mk}^{(2)},\\
b_{mk}^{(2)}(R)=\beta_{mk}^{(3)},\quad c_{mk}^{(2)}(R)=\gamma_{mk}^{(3)},\quad k\ge 1. &
\end{array}$$

This system is compatible according to uniqueness theorem, Putting the solution of the system (9) in (8), we'll find the solution of the problem (C). if $f^{(1)}(z)\in C^4(\partial\Omega)$, $f^{(j)}(z)\in C^3(\partial\Omega)$, $j=2,3$, then series (8) are absolutely and uniformly convergent in $\overline{\Omega^+}$ and $\overline{\Omega^-}$ areas.

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