Reports of Enlarged Session of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 20, №2, 2005

THE STATIONARY CURRENT OF THE MICROPOLAR VISCOUS INCOMPRESSIBLE FLUID OUTSIDE OF AN INFINITE PIPE

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Abstract

There obtained the general representation formula to the homogeneous system of stationary oscillation equations of the micropolar viscous incompressible fluid written in the from of one harmonic and two metaharmonic functions. Using this representation formula it is solved the exterior problem to circle when the limiting values of velocity and rotation are given on the boundary. The solution is obtained in the from of absolutely and uniformly convergent series.

Keywords and phrases: Micropolar, Viscosity, Green's Formula, Metaharmonic Function.

AMS subject classification (2000): 35J55; 74H25.

The homogeneous system of stationary oscillation equations of viscous incompressible fluid reeds as follows [1]

$$\begin{aligned}
&\operatorname{div} u(x) = 0,\\
&(\mu+\alpha)\Delta u(x) - 2\alpha \operatorname{rot}\omega(x) - \operatorname{grad} p(x) + \rho\sigma^2 u(x) = 0,\\
&(\nu+\beta)\Delta\omega(x) + 2\alpha \operatorname{rot} u(x) + (\mathcal{I}\sigma^2 - 4\alpha)\omega(x) = 0,
\end{aligned} \tag{1}$$

Where Δ is the laplace operator, $u(x) = (\ u_1(x),\ u_2(x)\ )^\top$ is the velocity, $p(x)$ and $\omega(x)$ are pressure function and rotation function respectively, $\sigma^2 = -i\tau$, τ is the oscillation (frequency) parameter, ρ is the density, I is the moment of inertia, α, β, μ, ν are positive physical parameters and

$$\operatorname{rot} u = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}, \qquad \operatorname{rot}\omega = \Big(-\frac{\partial \omega}{\partial x_2},\ \frac{\partial \omega}{\partial x_1}\Big)^\top,$$

here superscript $\top$ denotes transposition.

Let Ω^+ be the circle with the center in the origin and radius R
$\Omega^- = R^2 \setminus \overline{\Omega^+}$.

Problem. find a regular solution $U' = (\ u, \omega, p\ )^\top$ of the system (1) in the domain Ω^- satisfying the only one of the following two conditions on the boundary $\partial\Omega$

$$[u(z)]^- = f(z), \quad [\omega(z)]^- = f_3(z) \tag{2}$$

or

$$[H(\partial z,n)U'(z)]^- = f(z), \quad [\widetilde{T}(\partial z,n)\omega(z)]^- = f_3(z), \tag{3}$$

where $f(z) = (f_1(z), f_2(z))$, $f_j(z)$, $\quad j = 1,2,3$ are the given functions on the boundary $\partial\Omega$ $n(z) = (n_1(z), n_2(z))^\top$ is the outward unit normal vector to Ω^+ at the point $z \in \partial\Omega$, $s(z) = (-n_2(z), n_1(z))^\top$.

In the vicinity of infinity the vector $U'(x)$ satisfies the following conditions

$$\begin{aligned} &p(x) = o(|x|^{-1}) \qquad u_j(x) = O(1), \qquad \frac{\partial u_j(x)}{\partial x_k} = o(|x|^{-1}) \\ &\omega(x) = o(|x|^{-1}), \quad \frac{\partial \omega(x)}{\partial x_k} = o(|x|^{-1}), \quad k,j = 1,2. \end{aligned} \tag{4}$$

let us denote by $(I^\sigma)^-$ and $(II^\sigma)^-$ the problems (1)-(2) and (1)-(3) respectively. The stress vector has the following from

$$\begin{aligned} &H(\partial x,n)U'(x) = 2\mu\frac{\partial u}{\partial n(x)} - s(x)[(\mu-\alpha)\operatorname{rot} u + 2\alpha\omega] - n(x)p, \\ &\widetilde{T}(\partial x,n)\omega(x) = (\nu+\beta)\frac{\partial \omega}{\partial n(x)}. \end{aligned}$$

Theorem 1. *The problems $(I^\sigma)^-$ and $(II^\sigma)^-$ are uniquely solvable.*

Proof. The theorem will be proved if we show, that the corresponding homogeneous problems $(I^\sigma)^-$ and $(II^\sigma)^-$ $f_j = 0$, $\quad j = 1,2,3$ have only trivial solutions.

Green's formula has the following form in the domain Ω^- [3]

$$\int\limits_{\Omega^-} U \cdot M(\partial x)\overline{U}'\,dx = -\int\limits_{\partial\Omega} [U(z)]^- \cdot [T(\partial z,n)\overline{U}'(z)]^-\,ds - \int\limits_{\Omega^-} E(U,\overline{U})dx,. \tag{5}$$

where $U = (u,\omega)^\top$, $\quad E(u,\overline{u}) \geq 0$, [3]

$$M(\partial x)\overline{U'} = \begin{bmatrix} (\mu+\alpha)\Delta\overline{u} - 2\alpha\operatorname{rot}\overline{\omega} - \operatorname{grad}\overline{p} \\ (\nu+\beta)\Delta\overline{\omega} + 2\alpha\operatorname{rot}\overline{u} - 4\alpha\overline{\omega} \end{bmatrix}, \quad T(\partial x,n)\overline{U'} = \begin{bmatrix} H(\partial x,n)\overline{U'} \\ \widetilde{T}'(\partial x,n)\overline{\omega} \end{bmatrix},$$

using the equality $M(\partial x)\overline{U'} = \sigma^2[\ \rho\overline{u},\ I\overline{\omega}\ ]^\top$ and the boundary conditions of the problems $(I^\sigma)_0$ and $(II^\sigma)_0$, we have

$$\sigma^2\int\limits_{\Omega^-}[\ \rho|u|^2 + I|\omega|^2\ ]dx + \int\limits_{\Omega^-} E(U,\overline{U})dx = 0.$$

This last equality emplies that

$$\int\limits_{\Omega^-} |u(x)|^2\,dx = 0, \quad \int\limits_{\Omega^-} |\omega(x)|^2\,dx = 0.$$

and consequently $u(x)=0, \quad \omega(x)=0, \quad x\in\Omega^-$. If we use these equalities in (1),we have that $\operatorname{grad} p(x)=0, \quad x\in\Omega^-$ As $p(x)$ vanishes in the vicinity of infinity, we conclude that $p(x)=0, \quad x\in\Omega^-$. $\square$

Theorem 2. *The vector $U'=(u,\omega,p)^\top$ is the solution of the system (1) in the domain $\Omega^\pm$ if and only if can be represented in the following form*

$$u(x)=\operatorname{grad}\omega_3(x)+\sum_{j=1}^{2}k_j^2\operatorname{rot}\omega_j(x), \quad \omega(x)=\sum_{k=1}^{2}\omega_j(x), \quad p(x)=\rho\sigma^2\omega_3(x), \tag{6}$$

where

$$\Delta\omega_3(x)=0, \quad (\Delta+\lambda_j^2)\omega_j(x)=0, \quad j=1,2, \quad \lambda_1^2\cdot\lambda_2^2=\frac{\rho\sigma^2(I\sigma^2-4\alpha)}{(\mu+\alpha)(\nu+\beta)},$$

$$\lambda_1^2+\lambda_2^2=\frac{1}{(\mu+\alpha)(\nu+\beta)}[(\nu+\beta)\rho\sigma^2+(\mu+\alpha)(I\sigma^2-4\alpha)+4\alpha^2],$$

$$k_j^2=\frac{1}{2\alpha\rho\sigma^2}[-(\mu+\alpha)(\nu+\beta)\lambda_j^2+(\mu+\alpha)(I\sigma^2-4\alpha)+4\alpha^2].$$

Suppose that $Im\lambda_j>0, \quad j=1,2$. Let us solve the problem $(I^\sigma)^-$. We are looking for the solution of this problem in the form (6), where [4]

$$\begin{aligned}\omega_j(x)&=\sum_{k=0}^{\infty}h_k(\lambda_j r)(a_k^{(j)}\cos k\varphi+b_k^{(j)}\sin k\varphi), \quad j=1,2, \\ \omega_3(x)&=\sum_{k=0}^{\infty}\left(\frac{R}{r}\right)^k(a_k^{(j)}\cos k\varphi+b_k^{(j)}\sin k\varphi),\end{aligned}\tag{7}$$

here $a_k^{(j)}, \ b_k^{(j)}, \quad j=1,2,3$ are the sought for constants, $(r, \ \varphi)$ is the polar coordinates of the point x,

$$h_k(\lambda_j r)=\frac{H_k^{(1)}(\lambda_j r)}{H_k^{(1)}(\lambda_j R)}$$

where $H_k^{(1)}(x)$ is the Hankel function of first Kind.

Let us chainge the boundary condition (2)by its equivalent condition

$$[\ u_n(z)\ ]^-=F_1(z), \quad [\ u_s(z)\ ]^-=F_2(z), \quad [\ \omega(z)\ ]^-=F_3(z) \tag{8}$$

where

$$u_n=n_1u_1+n_2u_2, \quad u_s=-n_2u_1+n_1u_2, \quad F_1=n_1f_1+n_2f_2, \quad F_2=-n_2f_1+n_1f_2,$$

If we substitute the functions $\omega_j(x)$ from (7) into (6) we have

$$\begin{aligned} u_n(x) &= \sum_{k=0}^{\infty}(u_k^{(1)}(r)\cos k\varphi + v_k^{(1)}(r)\sin k\varphi),\\ u_s(x) &= \sum_{k=0}^{\infty}(u_k^{(2)}(r)\cos k\varphi + v_k^{(2)}(r)\sin k\varphi),\\ \omega(x) &= \sum_{k=0}^{\infty}\sum_{j=1}^{2}h_k(\lambda_j r)(a_k^{(j)}\cos k\varphi + b_k^{(j)}\sin k\varphi), \end{aligned} \tag{9}$$

$$\begin{bmatrix} u_k^{(1)}(r) \\ v_k^{(1)}(r) \end{bmatrix} = -\frac{k}{R}\Big(\frac{R}{r}\Big)^{k+1}\begin{bmatrix} a_k^{(3)} \\ b_k^{(3)} \end{bmatrix} - \sum_{j=1}^{2}\frac{kk_j^2}{r}h_k(\lambda_j r)\begin{bmatrix} b_k^{(j)} \\ -a_k^{(j)} \end{bmatrix},$$

$$\begin{bmatrix} -v_k^{(2)}(r) \\ u_k^{(2)}(r) \end{bmatrix} = \frac{k}{R}\Big(\frac{R}{r}\Big)^{k+1}\begin{bmatrix} a_k^{(3)} \\ b_k^{(3)} \end{bmatrix} - \sum_{j=1}^{2}k_j^2\frac{d}{dr}h_k(\lambda_j r)\begin{bmatrix} b_k^{(j)} \\ -a_k^{(j)} \end{bmatrix}.$$

Let us expand the functions $f_n(z)$, $f_s(z)$ and $f_3(z)$ into fourier series [3]

$$F_j(z) = \sum_{k=0}^{\infty}(\alpha_k^{(j)}\cos k\varphi + \beta_k^{(j)}\sin k\varphi), \quad j = 1, 2, 3, \tag{10}$$

Taking limit in the both side of the equality (9), as $x \to z \in \partial\Omega \quad (r \to R)$ and using the boundary condition (8) and equality (10), we obtain the system of algebraic equations to sought for constants

$$\begin{aligned} &\frac{k}{R}\begin{bmatrix} a_k^{(3)} \\ b_k^{(3)} \end{bmatrix} + \sum_{j=1}^{2}\frac{kk_j^2}{R}\begin{bmatrix} b_k^{(j)} \\ -a_k^{(j)} \end{bmatrix} = \begin{bmatrix} -\alpha_k^{(1)} \\ -\beta_k^{(1)} \end{bmatrix},\\ &\frac{k}{R}\begin{bmatrix} a_k^{(3)} \\ b_k^{(3)} \end{bmatrix} - \sum_{j=1}^{2}k_j^2\frac{d}{dR}h_k(\lambda_j R)\begin{bmatrix} b_k^{(j)} \\ -a_k^{(j)} \end{bmatrix} = \begin{bmatrix} -\beta_k^{(2)} \\ \alpha_k^{(2)} \end{bmatrix},\\ &\sum_{j=1}^{2}\begin{bmatrix} b_k^{(j)} \\ -a_k^{(j)} \end{bmatrix} = \begin{bmatrix} \beta_k^{(3)} \\ -\alpha_k^{(3)} \end{bmatrix}, \quad k \geq 0. \end{aligned} \tag{11}$$

Lemma. *To each trivial vector $U' = (u, \omega, p)^\top$,represented in the form (6) there corresponds the trivial vector $(\omega_1, \omega_2, \omega_3)^\top$ and vise-versa.*

Due to this lemma and the Theorem 1 the system (11) is consistent. Substituting the solution of this system in the representations (7) and (6), we obtain the solution of the problem $(I)^\sigma$. If we require that $f_j(z) \in C^2(\partial\Omega)$, $\quad j = 1, 2, 3$, then the series (9) and its derivative series will be absolutely and uniformly convergent in the domain $\overline{\Omega^-}$.

R E F E R E N C E S

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Received 25. IV. 2005; accepted 20. IX. 2005.