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ON A LINEAR INTEGRAL EQUATION OF THE THIRD KIND ARISING FROM THE NEUTRON TRANSPORT THEORY

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The aim of this paper is to study, in the class of Hölder functions, linear integral equations which frequently occurs when investigating many important problems of mathematical physics, namely in problems of the two-dimensional transport theory [1]. This equations have the form

$$\cos x\varphi(x) + \int_a^b K(x,y)\varphi(y)dy = f(x), \quad x \in (a,b), \tag{1}$$

where $\cos x$ vanishes at least once in the interval (a,b). By using the theory of singular integral equations, the necessary and sufficient conditions for the solvability of this equation under some assumptions on their kernels are given.

We assume that the kernel K of the equation (1) satisfies Hölders $\mathbf{H}$ condition, the right part $f \in \mathbf{H}^*$ (Muskhelishvili class) [2] and we look for solutions $\varphi \in \mathbf{H}^*$.

Let æ be the set of eigenvalues of the equation

$$(\cos x - z)\tau_z(x) + \int_a^b K(x,y)\tau_z(y)dy = 0, \quad x \in [a,b].$$

Denote

$$\omega(t,x) = \sum_{l=0}^{2n} \chi_e(t - l\pi)\chi_e(x - l\pi),$$

where $\chi_e(t)$ is the characteristic function of $[2a_e\pi, 2a_e\pi + \pi)$ $\;a_e = [\frac{a}{2\pi}]$ (entire part), $n = b'_e - a_e$, $\;b'_e = [\frac{b}{2\pi}] + 1$ and consider the following integral equation of the second kind

$$M(t,x) + \int_a^b \tilde{K}(t,x,y)M(t,y)dy = |\sin t|K(x,t), \quad t,x \in [a,b], \tag{2}$$

where

$$\tilde{K}(t,x,y) = \sum_{j=-n}^{n} \Big(\frac{K(x,y) - \chi(t^{(j)})K(x,t^{(j)})}{\cos y - \cos t}\omega(t^{(j)},y)$$
$$+ \frac{K(x,y) - \chi(\bar{t}^{(j)})K(x,\bar{t}^{(j)})}{\cos y - \cos t}\omega(\bar{t}^{(j)},y)\Big),$$

$t^{(j)} = t + 2j\pi$, $\;\bar{t}^{(j)} = 2m\pi - t + 2j\pi$, $\;m = b'_e + a_e$, and $\chi(t)$ is the characteristic function of $[a,b]$. Here t is a parameter.

We see that the kernel $\tilde{K}$ does not belong to that type of kernel which as a rule is called regular. In spite of this, all the fundamental Fredholm theorems are applicable to the equation (2), if they are stated in a suitable manner. This can be shown by reduction of (2) to a Fredholm equation with a bounded kernel (cf. [2]).

Further we consider the case when the following assumptions are satisfied:

A_1. æ is a finite set;

A_2. The homogeneous equation corresponding to (2) admits only zero solutions for any values of the parameter $t \in [a,b]$;

A_3. $K(x,y) = K(y,x)$.

These conditions are fulfilled for a sufficiently wide class of kernels K. Here they are introduced, having in mind the application to the original equations and in order to evoid some additional arguments.

We introduce the singular operator define by formula

$$L(u(\cdot))(x) := \sum_{j=-n}^{n} (\alpha_j(x)u(x^{(j)}) + \bar{\alpha}_j(x)u(\bar{x}^{(j)})) + \int_a^b \frac{M(t,x)}{\cos t - \cos x} u(t)dt,$$

where

$$\alpha_j(x) = |\sin x|\delta_{j,0} + \int_a^b \frac{\chi(x^{(j)})M(x^{(j)},y)}{\cos y - \cos x}\omega(x,y)dy,$$

$$\bar{\alpha}_j(x) = \int_a^b \frac{\chi(\bar{x}^{(j)})M(\bar{x}^{(j)},y)}{\cos y - \cos x}\omega(x,y)dy,$$

$x^{(j)} = x + 2j\pi$, $\bar{x}^{(j)} = 2m\pi - x + 2j\pi$, $\delta_{j,0}$ is the Kronecer symbol.

Theorem 1. *The equation*

$$L(u) = \psi_0 \tag{3}$$

is soluble if and only if $\psi_0 \in \mathbf{H}^*$ *on* (a,b) *satisfies the conditions*

$$\int_a^b \psi_0 \tau_{z_k} dx = 0, \quad z_k \in æ.$$

Provided these conditions are satisfied, the equation (3) has one and only one solution $u \in \mathbf{H}^*$ *on* (a,b).

Let S be the following integral operator

$$S(v(\cdot))(x) := \sum_{j=-n}^{n} (\chi(x^{(j)})\alpha_{-j}(x^{(j)})v(x^{(j)}) + \chi(\bar{x}^{(j)})\bar{\alpha}_j(\bar{x}^{(j)})v(\bar{x}^{(j)}))$$

$$+ \int_a^b \frac{M(x,t)}{\cos x - \cos t} v(t)dt, \quad x \in (a,b), \quad v \in H^*.$$

Denote

$$\beta_j(x) = \xi_j(x)$$

$$+\pi^2\chi(x^{(j)}) \sum_{l=j-n}^{j+n} (\chi(x^{(l)})Q(x,x^{(l)})Q(x^{(j)},x^{(l)}) + \chi(\bar{x}^{(l)})Q(x,\bar{x}^{(l)})Q(x^{(j)},\bar{x}^{(l)})),$$

$$\bar{\beta}_j(x) = \bar{\xi}_j(x)$$

$$+\pi^2\chi(\bar{x}^{(j)})\sum_{l=j-n}^{j+n}(\chi(x^{(l)})Q(x,x^{(l)})Q(\bar{x}^{(j)},x^{(l)})+\chi(\bar{x}^{(l)})Q(x,\bar{x}^{(l)})Q(\bar{x}^{(j)},\bar{x}^{(l)})),$$

where

$$\xi_j(x)=\sum_{l=j-n}^{j+n}(\chi(x^{(l)})\alpha_{-l}(x^{(l)})\alpha_{j-l}(x^{(l)})+\chi(\bar{x}^{(-l)})\bar{\alpha}_{-l}(\bar{x}^{(-l)})\bar{\alpha}_{j-l}(\bar{x}^{(-l)})),$$

$$\bar{\xi}_j(x)=\sum_{l=j-n}^{j+n}(\chi(x^{(-l)})\alpha_l(x^{(-l)})\bar{\alpha}_{j-l}(x^{(-l)})+\chi(\bar{x}^{(l)})\bar{\alpha}_l(\bar{x}^{(l)})\alpha_{j-l}(\bar{x}^{(l)})),$$

Q is defined from the equality $M(t,x)=\mid\sin t\mid Q(t,x)$.

Theorem 2. *The composition SL contains no singular part and the following equality holds*

$$S(L(u))(x)=\sum_{j=-n}^{n}(\beta_j(x)u(x^{(j)})+\bar{\beta}_j(x)u(\bar{x}^{(j)})).$$

Let T be the following integral operator

$$T(v(\cdot))(x):=\frac{1}{\mid\mathbf{B}(x)\mid}\sum_{j=-n}^{n}(\sigma_{0j}^{(1,1)}(x)S^*(v)(x^{(j)})+\sigma_{0j}^{(1,2)}(x)S^*(v)(\bar{x}^{(j)})),\quad x\in(a,b),$$

where $\sigma_{0j}^{(r,s)}(x)$ is algebraic adjunct of $\beta_{j0}^{(s,r)}(x)$ in $\mid\mathbf{B}(x)\mid$, $\mathbf{B}(x)=\parallel B_{rs}(x)\parallel$ $(r,s=1,2)$ is the square block matrix, where $B_{rs}=\parallel b_{ij}^{(r,s)}(x)\parallel$ $(i,j=\overline{-n,n})$ is the square matrix with elements:

$$b_{ij}^{(1,1)}=\tilde{\chi}\beta_{j-i}(x^{(i)}),\ b_{ij}^{(1,2)}=\tilde{\chi}\bar{\beta}_{i-j}(x^{(i)}),$$

$$b_{ij}^{(2,1)}=\tilde{\chi}\bar{\beta}_{j-i}(\bar{x}^{(-i)}),\ b_{ij}^{(2,2)}=\tilde{\chi}\beta_{i-j}(\bar{x}^{(-i)}).$$

Here $\tilde{\chi}=\chi(x^{(i-j)})$.

Theorem 3. *Let $K\in\mathbf{H}$ be such that the assumptions A_i, $(i=1,2,3)$ are fulfilled and $f\in\mathbf{H}^*$. equation (1) is soluble, if and only if the function f satisfies the conditions*

$$T(f)(\frac{\pi}{2}+(2a_e+l)\pi)=0,\quad l=\overline{0,2n}.$$

Provided these conditions are satisfied, equation (1) has one and only one solution $\varphi(x)\in\mathbf{H}^*$ on (a,b) and this solution may be written as

$$\varphi(x)=\sum_k\frac{\tau_{z_k}(x)}{z_kN_{z_k}}\int_a^b f(y)\tau_{z_k}(y)dy+L\Big(\frac{1}{\cos(\cdot)}T(f)(\cdot)\Big)(x),$$

where

$$N_{z_k}=\int_a^b\tau_{z_k}^2dx,\quad z_k\in æ.$$

R E F E R E N C E S

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