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ELASTIC BODY IN THE SCALAR FIELD

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Abstract

It is considered the contact problem for domains bounded by spherical surfaces. Uniqueness theorem of the problem is proved. The solutions are represented by absolutely and uniformly convergent series.

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We denote by Ω^+ the ball bounded by the sphere $\partial\Omega$ centered at the origin and with radius R, i.e. $\Omega^+ = \{x : \ x \in \mathbb{R}^3, \ |x| < R\}$, $\partial\Omega = \{x : \ x \in \mathbb{R}^3, \ |x| = R\}$. Let $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega}^+$. Let us consider the following problem:

Problem (A). Find a regular vector $U(x) = (u(x), \omega(x))$ and a scalar function $v(x)$ in Ω^+ and Ω^-, respectively, which satisfy:
a) the differential equations [2]

$$\begin{aligned} &(\mu+\alpha)\Delta u(x) + (\lambda+\mu-\alpha)\,\mathrm{grad}\,\mathrm{div}\,u(x) + 2\alpha\,\mathrm{rot}\,\omega(x) = 0,\\ &(\nu+\beta)\Delta\omega(x) + (\varepsilon+\nu-\beta)\,\mathrm{grad}\,\mathrm{div}\,\omega(x) + 2\alpha\,\mathrm{rot}\,u(x) - 4\alpha\omega(x) = 0, \ x \in \Omega^+, \end{aligned} \tag{1}$$

$$\Delta v(x) = 0, \quad x \in \Omega^-; \tag{2}$$

b) the contact and boundary conditions on $\partial\Omega$

$$\begin{aligned} &[n(z)u(z)]^+ - d_1\Big[\frac{\partial v(z)}{\partial n(z)}\Big]^- = f_4(z),\\ &[H(\partial z, n)U(z)]^+ - d_2 n(z)[v(z)]^- = f^{(1)}(z),\\ &[\widetilde{T}(\partial z, n)\omega(z)]^+ = f^{(2)}(z); \end{aligned} \tag{3}$$

c) while in the vicinity of the infinity the function $v(x)$ meets the following asymptotic relations

$$v(x) = O(|x|^{-1}), \quad \frac{\partial v(x)}{\partial x_j} = o(|x|^{-1}), \quad j = 1,2,3, \tag{4}$$

where $u(x) = (u_1(x), u_2(x), u_3(x))$ is the displacement vector, $\omega(x) = (\omega_1(x), \omega_2(x), \omega_3(x))$ vector of rotation, $f^{(j)} = (f_1^{(j)}, f_2^{(j)}, f_3^{(j)})$, $f_4(z)$, $f_k^{(j)}(z)$, $k = 1,2,3$, $j = 1,2$, are given functions on $\partial\Omega$. $n(z)$ is the outward (to Ω^+) unit normal vector at the point $z \in \partial\Omega$,

$$H(\partial x, n)U(x) = 2\mu \frac{\partial u(x)}{\partial n(x)} + \lambda n(x) \operatorname{div} u(x) + (\mu - \alpha)[n(x) \times \operatorname{rot} u(x)]$$
$$+2\alpha[n(x) \times \omega(x)],$$
$$\widetilde{T}(\partial x, n)\omega(x) = 2\nu \frac{\partial \omega(x)}{\partial n(x)} + \varepsilon n(x) \operatorname{div} \omega(x) + (\nu - \beta)[n(x) \times \operatorname{rot} \omega(x)],$$

λ, μ, ν, α, β, ε, d_1, d_2 are the constants, which satisfy the conditions: $\mu > 0$, $\alpha > 0$, $\nu > 0$, $\beta > 0$, $3\lambda + 2\mu > 0$, $3\varepsilon + 2\nu > 0$, $d_1 d_2 > 0$.

Theorem. *The general solution of the problem* $(A)_0$ $(f_4 = 0,\ f^{(j)} = 0,\ j = 1, 2)$ *is*

$$u(x) = [a \times x],\ \ \omega(x) = a,\ \ x \in \Omega^+, \quad v(x) = 0,\ \ x \in \Omega^-.$$

Proof. We have Green's formulas for (1),(2) system in domains $\Omega^\pm$ [2]

$$\int_{\Omega^+} U(x) \cdot M(\partial x)U(x)\, dx = \int_{\partial\Omega} [u(z) \cdot H(\partial z, n)U(z)$$
$$+\omega(z) \cdot \widetilde{T}(\partial z, n)\omega(z)]^+\, ds - \int_{\Omega^+} E(U, U)\, dx, \tag{5}$$
$$\int_{\Omega^-} v(x)\Delta v(x)\, dx = -\int_{\partial\Omega} \Big[v(z)\frac{\partial v(z)}{\partial n(z)}\Big]^-\, ds - \int_{\Omega^-} (\operatorname{grad} v(x))^2\, dx,$$

where

$$E(U, U) = \frac{3\lambda + 2\mu}{3} (\operatorname{div} u)^2 + \frac{3\varepsilon + 2\nu}{3} (\operatorname{div} \omega)^2 + \frac{\mu}{2} \sum_{i,j=1}^{3} \Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij} \operatorname{div} u\Big)^2$$
$$+\frac{\nu}{2} \sum_{i,j=1}^{3} \Big(\frac{\partial \omega_i}{\partial x_j} + \frac{\partial \omega_j}{\partial x_i} - \frac{2}{3}\delta_{ij} \operatorname{div} \omega\Big)^2 + \frac{\alpha}{2} \sum_{i,j=1}^{3} \Big(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j} + 2\sum_{k=1}^{3} \varepsilon_{kji}\omega_k\Big)^2$$
$$+\frac{\beta}{2} \sum_{i,j=1}^{3} \Big(\frac{\partial \omega_j}{\partial x_i} - \frac{\partial \omega_i}{\partial x_j}\Big)^2,$$

$M(\partial x)U(x)$ is the left part of the system (1).

Taking into account the conditions of the Problem$(A)_0$. from (5) we have

$$E(U, U) = 0, \quad \operatorname{grad} v(x) = 0.$$

The solutions of this equations has the form [2]

$$u(x) = [a \times x] + b,\ \ \omega(x) = a,\ \ x \in \Omega^+, \quad v(x) = c,\ \ x \in \Omega^-,$$

where a and b are three-dimensional vectors, c is a constant.

From the conditions (4) and boundary conditions of problem $(A)_0$ we have

$$u(x)=[a\times x],\ \ \omega(x)=a,\ \ x\in\Omega^+,\quad v(x)=0,\ \ x\in\Omega^-.\square$$

Let us seek the solutions of the formulated problem (A) in the form [3]

$$\begin{aligned}
u(x)&=\operatorname{grad}\Phi_1(x)-a\operatorname{grad}r^2\Big(r\frac{\partial}{\partial r}+1\Big)\Phi_2(x)+\operatorname{rot}\operatorname{rot}(xr^2\Phi_2(x))+\\
&\quad+\operatorname{rot}(x\Phi_3(x))+2\alpha[\operatorname{rot}\operatorname{rot}(x\Phi_5(x))+\operatorname{rot}(x\Phi_6(x))],\\
\omega(x)&=\operatorname{grad}\Phi_4(x)-\operatorname{rot}\Big[x\Big(2r\frac{\partial}{\partial r}+3\Big)\Phi_2(x)\Big]+\frac{1}{2}\operatorname{rot}\operatorname{rot}(x\Phi_3(x))\\
&\quad-(\mu+\alpha)[\lambda_2^2\operatorname{rot}(x\Phi_5(x))-\operatorname{rot}\operatorname{rot}(x\Phi_6(x))],
\end{aligned}\tag{6}$$

$$v(x)=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big(\frac{R}{r}\Big)^{k+1}Y_k^{(m)}(\vartheta,\varphi)B_{mk},\tag{7}$$

where B_{mk} is sought for constant, $x=(x_1,x_2,x_3)$, $r=|x|$, $r\frac{\partial}{\partial r}=(x\cdot\operatorname{grad})$, $\Delta\Phi_j(x)=0$, $j=1,2,3$, $(\Delta-\lambda_1^2)\Phi_4(x)=0$, $(\Delta-\lambda_2^2)\Phi_j(x)=0$, $j=5,6$, $a=\mu(\lambda+2\mu)^{-1}$, $\lambda_1^2=4\alpha(\varepsilon+2\nu)^{-1}$, $\lambda_2^2=4\alpha\mu[(\nu+\beta)(\mu+\alpha)]^{-1}$,

$$Y_k^{(m)}(\vartheta,\varphi)=\sqrt{\frac{2k+1}{4\pi}\cdot\frac{(k-m)!}{(k+m)!}}P_k^{(m)}(\cos\vartheta)e^{im\varphi}.$$

We look for functions $\Phi_j(x)$, $j=1,2,\dots,6$, in the following form

$$\begin{aligned}
\Phi_j(x)&=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big(\frac{r}{R}\Big)^{k}Y_k^{(m)}(\vartheta,\varphi)A_{mk}^{(j)},\ \ j=1,2,3,\\
\Phi_4(x)&=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}g_k(\lambda_1,r)Y_k^{(m)}(\vartheta,\varphi)A_{mk}^{(4)},\\
\Phi_j(x)&=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}g_k(\lambda_2,r)Y_k^{(m)}(\vartheta,\varphi)A_{mk}^{(j)},\ \ j=5,6,
\end{aligned}\tag{8}$$

where

$$g_k(\lambda_jr)=\sqrt{\frac{R}{r}}\,\frac{I_{k+\frac12}(\lambda_jr)}{I_{k+\frac12}(\lambda_jR)},\ \ j=1,2.$$

Upon substitution of the $\Phi_j(x)$ from (8) into the representations $u(x)$ and $\omega(x)$, we have

$$\begin{aligned}
u(x)&=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big[u_{mk}^{(1)}(r)X_{mk}(\vartheta,\varphi)+\sqrt{k(k+1)}\Big(v_{mk}^{(1)}(r)Y_{mk}(\vartheta,\varphi)\\
&\quad+w_{mk}^{(1)}(r)Z_{mk}(\vartheta,\varphi)\Big)\Big],\\
\omega(x)&=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big[u_{mk}^{(2)}(r)X_{mk}(\vartheta,\varphi)+\sqrt{k(k+1)}\Big(v_{mk}^{(2)}(r)Y_{mk}(\vartheta,\varphi)\\
&\quad+w_{mk}^{(2)}(r)Z_{mk}(\vartheta,\varphi)\Big)\Big]
\end{aligned}\tag{9}$$

The stress vectors has the following form

$$H(\partial x, n)U(x) = \sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big[a_{mk}^{(1)}(r)X_{mk}(\vartheta,\varphi) + \sqrt{k(k+1)}\Big(b_{mk}^{(1)}(r)Y_{mk}(\vartheta,\varphi)$$
$$+c_{mk}^{(1)}(r)Z_{mk}(\vartheta,\varphi)\Big)\Big],$$
$$\widetilde{T}(\partial x, n)\omega(x) = \sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big[a_{mk}^{(2)}(r)X_{mk}(\vartheta,\varphi) + \sqrt{k(k+1)}\Big(b_{mk}^{(2)}(r)Y_{mk}(\vartheta,\varphi) \qquad (10)$$
$$+c_{mk}^{(2)}(r)Z_{mk}(\vartheta,\varphi)\Big)\Big]$$

where $u_{mk}^{(j)}(z)$, $v_{mk}^{(j)}(z)$,...,$c_{mk}^{(j)}(z)$, $j = 1,2$ are functions of-r, $X_{mk}(\vartheta,\varphi)$, $Y_{mk}(\vartheta,\varphi)$, $Z_{mk}(\vartheta,\varphi)$ are orthonormal vectors from $L_2(\Sigma_1)$. [1]

Let us assume that the function $f_4(z)$ and the vector $f^{(j)}(z)$, $j = 1,2$, satisfy the sufficient smoothness conditions, that allow us to represent them into a Fourier-Laplas series

$$f^{(j)}(z) = \sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big[\alpha_{mk}^{(j)}(r)X_{mk}(\vartheta,\varphi) + \sqrt{k(k+1)}\Big(\beta_{mk}^{(j)}(r)Y_{mk}(\vartheta,\varphi)$$
$$+\gamma_{mk}^{(j)}(r)Z_{mk}(\vartheta,\varphi)\Big)\Big], \quad j = 1,2, \qquad (11)$$
$$f_4(z) = \sum_{k=0}^{\infty}\sum_{m=-k}^{k}\alpha_{mk}^{(4)}Y_k^{(m)}(\vartheta,\varphi).$$

If in (7), (9), (10), we pass to the limit as $x \to z \in \partial\Omega$ $(r \to R)$, with the help of (11) and the contact and boundary conditions (3), we obtain the system of linear algebraic equations for the sought for constants $A_{mk}^{(j)}$, B_{mk},

$$\begin{aligned} &u_{mk}^{(1)}(R) - d_1(k+1)R^{-1}B_{mk} = \alpha_{mk}^{(4)}, \;\; a_{mk}^{(1)}(R) - d_2B_{mk} = \alpha_{mk}^{(1)}, \;\; k \ge 0, \\ &b_{mk}^{(j)}(R) = \beta_{mk}^{(j)}, \;\; c_{mk}^{(j)}(R) = \gamma_{mk}^{(j)}, \;\; k \ge 1, \;\; j = 1,2, \;\; a_{mk}^{(2)}(R) = \alpha_{mk}^{(2)}, \;\; k \ge 0. \end{aligned} \qquad (12)$$

The condition

$$\int_{\partial\Omega}[z \times f^{(1)}(z) + f^{(2)}(z)]ds = 0$$

is necessary and sufficient for the above problem to be solvable.

Substituting of functions $f^{(j)}(z)$, $j = 1,2$, into the last formulae, we see that the coefficient $\beta_{mk}^{(2)}$ and $\gamma_{mk}^{(1)}$ satisfy the following condition

$$\beta_{mk}^{(2)} + R\gamma_{mk}^{(1)} = 0, \quad m = 0, \pm 1. \qquad (13)$$

Proceeding from the uniqueness theorem for the Problem (A) and condition (13), we can infer to (12) system is solvable and the constant $A_{m1}^{(3)}$ is not defined. This is natural since the solution of the Problem (A) is defined within the vector

$$u(x) = [a \times x], \;\; \omega(x) = a, \;\; v(x) = 0.$$

If $f^{(j)}(z) \in C^3(\partial\Omega)$, $j = 1, 2$, $f_4(z) \in C^4(\partial\Omega)$, then the obtained series (7), (9), (10) are absolutely and uniformly convergent.

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