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THE METHOD OF A SMALL PARAMETER FOR THE NON-SHALLOW SHELLS

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1. There exist several methods of reduction of the three-dimensional problems to the two-dimensional ones (Khirchhoff-Love, E. Reissner, A. Green, W. Koiter, P. Nagdi, A. Goldenveizer, I. Vorovoch, I. Vekua, etc).

Under thin and shallow shells I. Vekua meant the three-dimensional shell-type elastic bodies satisfying the conditions

$$1-k_\alpha x_3 \equiv 1 \ \ (\alpha,\ \beta=1,2), \ \ -h \leq x_3 \leq h \tag{$*$}$$

(h-semithickness of shell Ω), which means that geometry of the shell does't vary in thickness and therefore such kind of shells are usually called the shells with non-varing geometry. Here k_1 and k_2 are principal curvature of the midsurface S.

In the sequal, under non-shallow shells will be meant elastic bodies free from the assumption of the type $(*)$, or more exactly the bodies with the conditions

$$1-k_\alpha x_3 \neq 1, \ \ \mid hk_\alpha \mid \leq q<1.$$

Such kind of shells are called shells with varying in thickness geometry, or non-shallow shells.

Now following I. Vekua we obtain the system of two-dimensional equations, which is lines of curvature coordinates written as:

$$a)\begin{cases} h\nabla_\alpha \overset{(m)}{\sigma}{}^{\alpha_1} - \varepsilon k_1 R \overset{(m)}{\sigma}{}^{13} - (2m+1)\left(\overset{(m-1)}{\sigma}{}^{31} + \overset{(m-3)}{\sigma}{}^{31} + ...\right) + h\overset{(m)}{F}{}^{1} = 0, \\ h\nabla_\alpha \overset{(m)}{\sigma}{}^{\alpha_2} - \varepsilon k_2 R \overset{(m)}{\sigma}{}^{23} - (2m+1)\left(\overset{(m-1)}{\sigma}{}^{32} + \overset{(m-3)}{\sigma}{}^{32} + ...\right) + h\overset{(m)}{F}{}^{2} = 0, \\ h\nabla_\alpha \overset{(m)}{\sigma}{}^{\alpha 3} + \varepsilon R(k_1 \overset{(m)}{\sigma}{}^{1}_{1} + k_2 \overset{(m)}{\sigma}{}^{2}_{2}) - (2m+1)\left(\overset{(m-1)}{\sigma}{}^{33} + \overset{(m-3)}{\sigma}{}^{33} + ...\right) \\ \quad +h\overset{(m)}{F}{}^{3} = 0, \ \ \left(\overset{(k)}{\sigma}{}^{ij} = 0, \ \ k<0, \ \ i,j=1,2,3\right) \ \ (m=0,1,...N) \end{cases} \tag{1}$$

where $\varepsilon = \frac{h}{R}$ is a small parameter, R-characteristic radius of curvature, then

$$\begin{aligned} \overset{(m)}{\mathbf{F}} &= \overset{(m)}{\Phi} + \frac{2m+1}{h}\Big[(1-\varepsilon k_1 R)(1-\varepsilon k_2 R)\overset{(+)}{\boldsymbol{\sigma}}{}^{3} \\ &- (-1)^m(1+\varepsilon k_1 R)(1+\varepsilon k_2 R)\overset{(-)}{\boldsymbol{\sigma}}{}^{3}\Big], \end{aligned}$$

b) Hooke's law

$$\begin{cases} h\overset{(m)}{\sigma}{}^{\alpha\beta} = \sum\limits_{s=0}^{\infty}\left\{\overset{(m,s)}{I}{}^{\alpha\gamma}_{\alpha_1\gamma_1}\left[\lambda(h\nabla_\gamma \overset{(s)}{\boldsymbol{U}}{}^{\gamma_1} - \varepsilon b^{\gamma_1}_{\gamma} R\overset{(s)}{\boldsymbol{U}}{}_3)a^{\alpha_1\beta} + \mu(h\nabla_\gamma \overset{(s)}{\boldsymbol{U}}{}^{\alpha_1},\right.\right. \\ \left.\left. - \varepsilon b^{\alpha_1}_{\gamma} R\overset{(s)}{\boldsymbol{U}}{}_3)a^{\beta\gamma_1} + \mu(h\nabla_\gamma \overset{(s)}{\boldsymbol{U}}{}^{\beta} - \varepsilon b^{\beta}_{\gamma} R\overset{(s)}{\boldsymbol{U}}{}_3)a^{\alpha_1\gamma_1}\right] + \lambda\overset{(m,s)}{I}{}^{3\alpha}_{3\alpha_1}\overset{(s)}{\boldsymbol{U}}{}'_3 a^{\alpha_1\beta}\right\}, \\ h\overset{(m)}{\sigma}{}^{\alpha 3} = \mu\sum\limits_{s=0}^{\infty}\left[\overset{(m,s)}{I}{}^{\alpha\gamma}_{\alpha_1\gamma_1}(h\nabla_\gamma \overset{(s)}{\boldsymbol{U}}{}_3 + \varepsilon b^{\eta}_{\gamma} R\overset{(s)}{\boldsymbol{U}}{}_\eta)a^{\alpha_1\gamma_1} + \overset{(m,s)}{I}{}^{3\alpha}_{3\alpha_1}\overset{(s)}{\boldsymbol{U}}{}'^{\alpha_1}\right], \\ h\overset{(m)}{\sigma}{}^{3\alpha} = \mu\sum\limits_{s=0}^{\infty}\left[\overset{(m,s)}{I}{}^{3\alpha}_{3\alpha_1}(h\nabla_\gamma \overset{(s)}{\boldsymbol{U}}{}_3 + \varepsilon b^{\eta}_{\gamma} R\overset{(s)}{\boldsymbol{U}}{}_\eta)a^{\alpha_1\gamma} + \overset{(m,s)}{I}{}^{33}_{33}\overset{(s)}{\boldsymbol{U}}{}'^{\alpha_1}\right], \\ h\overset{(m)}{\sigma}{}^{33} = \sum\limits_{s=0}^{\infty}\left[\lambda\overset{(m,s)}{I}{}^{3\gamma}_{3\gamma_1}(h\nabla_\gamma \overset{(s)}{\boldsymbol{U}}{}^{\gamma_1} - \varepsilon b^{\gamma_1}_{\gamma} R\overset{(s)}{\boldsymbol{U}}{}_3) + (\lambda+2\mu)\overset{(m,s)}{I}{}^{33}_{33}\overset{(s)}{\boldsymbol{U}}{}'_3\right] \end{cases} \tag{2}$$

where $b^1_1 = k_1,\ \ b^2_2 = k_2,\ \ b^1_2 = b^2_1 = 0,$

$$\overset{(s)}{\boldsymbol{u}}{}' = (2s+1)\left(\overset{(s+1)}{\boldsymbol{u}} + \overset{(s+3)}{\boldsymbol{u}} + ...\right),$$

$$\overset{(m,s)}{I}{}^{\alpha\alpha}_{\alpha\alpha} = \frac{2m+1}{2h}\int\limits_{-h}^{h}\frac{1-k_\beta x_3}{1-k_\alpha x_3}p_m p_s dx_3 = \frac{k_\beta}{k_\alpha}\delta^{ms} + \frac{2m+1}{2k_\alpha h}\frac{k_\alpha - k_\beta}{k_\alpha}P_m\left(\frac{1}{k_\alpha h}\right)Q_s\left(\frac{1}{k_\alpha h}\right)$$

$$= \frac{k_\beta}{k_\alpha}\delta^{ms} + \frac{k_\alpha - k_\beta}{k_\alpha}\sum_{p=0}^{[\frac{m}{2}]}\sum_{q=0}^{\infty}\overset{(m,s)}{M}{}_{pq}(\varepsilon R k_\alpha)^{s-m+2(p+q)},\ \ (m \le s;\ \ \alpha \neq \beta).$$

Here P_m and Q_s - are Legendre polynomial and functions of the second kind, then

$$\begin{aligned} \overset{(m,s)}{M}{}_{pq} &= 2^{s-m}\frac{(-1)^p}{p!}\frac{(2m-2p)!}{(m-p)!(m-2p)!}\frac{(s+q)!(s+2q)!}{(2s+2q+1)!}, \quad (m \le s), \\ \overset{(m,s)}{I}{}^{\alpha\beta}_{\alpha\beta} &= \overset{(m,s)}{I}{}^{\beta\alpha}_{\beta\alpha} = \delta_{ms}, \quad \overset{(m,s)}{I}{}^{\alpha\cdot}_{\beta\cdot} = 0, \ \ (\alpha \neq \beta), \\ \overset{(m,s)}{I}{}^{\alpha 3}_{\alpha 3} &= \overset{(m,s)}{I}{}^{3\alpha}_{3\alpha} = \frac{2m+1}{2h}\int\limits_{-h}^{h}(1-k_\beta x_3)p_m p_s dx_3 \\ &= \delta^{ms} - \varepsilon k_\beta R\left(\frac{m}{2m-1}\delta^{s-1}_m + \frac{m+1}{2m+3}\delta^{s+1}_m\right) \quad (\alpha \neq \beta), \\ \overset{(m,s)}{I}{}^{33}_{33} &= \frac{2m+1}{2h}\int\limits_{-h}^{h}\sqrt{\frac{g}{a}}p_m p_s dx_3 = \delta_{ms} - 2H\varepsilon R\left(\frac{m}{2m-1}\delta^{s-1}_m + \frac{m+1}{2m+3}\delta^{s+1}_m\right) \\ &+ \varepsilon^2 K R^2\left(\frac{m(m-1)}{(2m+1)(2m-3)}\delta^{s+1}_m + \frac{1}{2m+1}\left(\frac{(m+1)^2}{2m+3} + \frac{m^2}{2m-1}\right)\delta^s_m\right. \\ &+ \left.\frac{(m+1)(m+2)}{(2m+3)(2m+5)}\delta^{s-2}_m\right) \end{aligned}$$

$$(2H = k_1 + k_2, \ \ K = k_1 k_2).$$

c) Boundary conditions of the lateral contour of midsurface S:

$$\begin{aligned} I)\ & \overset{(m)}{\sigma}_{(\mathbf{ll})} = \overset{(m)}{\sigma}{}^{\alpha\beta} l_\alpha l_\beta = f_1, \quad \overset{(m)}{\sigma}_{(\mathbf{ls})} = \overset{(m)}{\sigma}{}^{\alpha\beta} l_\alpha s_\beta = f_2, \\ & \overset{(m)}{\sigma}_{(\mathbf{ln})} = \overset{(m)}{\sigma}{}^{\alpha 3} l_\alpha = f_3, \\ II)\ & \overset{(m)}{U}_{(\mathbf{l})} = \overset{(m)}{U}{}^{\alpha} l_\alpha = g_1, \quad \overset{(m)}{U}_{(\mathbf{s})} = \overset{(m)}{U}{}^{\alpha} l_\alpha = g_2, \quad \overset{(m)}{U}_{3} = g_3. \end{aligned} \tag{3}$$

$$(\mathbf{l} \times \mathbf{s} = \mathbf{n}, \ \ l_\alpha = \mathbf{l} \cdot \mathbf{r}_\alpha, \ \ s_\alpha = \mathbf{s} \cdot \mathbf{r}_\alpha)$$

2. Approximations of order $N = 0$ **and** $N = 1$. By introducing the following notations

$$\left(\overset{(0)}{\sigma}{}^{ij}, \ \overset{(0)}{\boldsymbol{U}}, \ \overset{(0)}{F}{}^{i} \right) = \left(T^{ij}, \ \mathbf{u}, \ X^i \right), \quad \overset{(0,0)}{I}{}^{\cdot\,\cdot}_{\cdot\,\cdot} = I^{\cdot\,\cdot}_{\cdot\,\cdot}$$

for approximation $N = 0$, we obtain

a) Equation of equilibrium

$$\begin{cases} h(\nabla_1 T^{1\alpha} + \nabla_2 T^{2\alpha}) - \varepsilon k_1 R T^{\alpha 3} + h X^\alpha = 0 (\alpha = 1, 2), \\ h(\nabla_1 T^{13} + \nabla_2 T^{23}) + \varepsilon R (k_1 T_1^1 + k_2 T_2^2) + h X^3 = 0. \end{cases}$$

b) Hooke's law

$$\begin{cases} hT^{\alpha\beta} = I^{\alpha\gamma}_{\alpha_1\gamma_1} \left[\lambda (h \nabla_\gamma u^{\gamma_1} - \varepsilon b^{\gamma_1}_\gamma R u_3) a^{\alpha_1\beta} + \mu (h \nabla_\gamma u^{\alpha_1} - \varepsilon b^{\alpha_1}_\gamma R u_3) a^{\beta\gamma_1} \right. \\ \qquad \left. + \mu (h \nabla_\gamma u^{\beta} - \varepsilon b^{\beta}_\gamma R u_3) a^{\alpha_1\gamma_1} \right], \\ hT^{\alpha 3} = \mu I^{\alpha\gamma}_{\alpha_1\gamma_1} (h \nabla_\gamma u_3 + \varepsilon b^\eta_\gamma R u_\eta) a^{\alpha_1\gamma_1}, \\ I^{\alpha\alpha}_{\alpha\alpha} = 1 + \varepsilon^2 \dfrac{k_\alpha^2 - k_\alpha k_\beta}{3} R^2 + \cdots + \varepsilon^{2p} \dfrac{k_\alpha^{2p} - k_\alpha^{2p-1} k_\beta}{2p+1} R^{2p} + \cdots \end{cases}$$

$$I^{\alpha\beta}_{\alpha\beta} = I^{\beta\alpha}_{\beta\alpha} = 1, \ \ I^{33}_{33} = 1, (\alpha \neq \beta).$$

To determine the components of the displacement vector and stress tenzor we shell use expansions with respect to the small parameter ε:

$$\left(T^{ij}, \ \mathbf{u}, \ X^i \right) = \sum_{n=0}^{\infty} \left(\overset{(n)}{T}{}^{ij}, \ \overset{(n)}{u}, \ \overset{(n)}{X}{}^{i} \right) \varepsilon^n$$

and then equate to zero the factors ε^n, we have

$$\begin{cases} \mu \nabla_\alpha \nabla^\alpha \overset{(n)}{u}_\beta + (\lambda + \mu) \partial_\beta \overset{(n)}{\theta} = \overset{(n)}{Y}_\beta (\overset{0}{\boldsymbol{u}}, ..., \overset{(n-1)}{\boldsymbol{u}}) \ \ (\beta = 1, 2) \\ \mu \nabla_\alpha \nabla^\alpha \overset{(n)}{u}_3 = \overset{(n)}{Y}_3 (\overset{0}{\boldsymbol{u}}, ..., \overset{(n-1)}{\boldsymbol{u}}) \ \ (\overset{(n)}{\theta} = \nabla_\alpha \overset{(n)}{u}{}^{\alpha}). \end{cases}$$

$$(n = 0, 1, 2, ...,)$$

The general solution of the system is expressed by three harmonic functions on the midsurface S, and it ensure the satisfaction of three given physical or kinematic conditions [1].

For approximate of order $N=1$ we have the following systems equation of equilibrium

$$\begin{cases} \mu h \nabla_\alpha \nabla^\alpha \overset{(n)}{u}_\beta + (\lambda+\mu) h \partial_\beta \overset{(n)}{\theta} + 2\lambda \partial_\beta \overset{(n)}{v}_3 = \overset{(n)}{A}_\beta (\overset{0}{\boldsymbol{u}}, \overset{(0)}{\boldsymbol{v}}, ..., \overset{(n-1)}{\boldsymbol{u}}, \overset{(n-1)}{\boldsymbol{v}}), \\ \mu \left(h \nabla_\alpha \nabla^\alpha \overset{(n)}{u}_3 + \overset{(n)}{\rho} \right) = \overset{(n)}{A}_3 (\overset{0}{\boldsymbol{u}}, \overset{(0)}{\boldsymbol{v}}, ..., \overset{(n-1)}{\boldsymbol{u}}, \overset{(n-1)}{\boldsymbol{v}}), \\ \mu h^2 \nabla_\alpha \nabla^\alpha \overset{(n)}{v}_\beta + (\lambda+\mu) h^2 \partial_\beta \overset{(n)}{\rho} - 3\mu \left(h \partial_\beta \overset{(n)}{u}_3 + \overset{(n)}{v}_\beta \right) = \overset{(n)}{B}_\beta (\overset{0}{\boldsymbol{u}}, \overset{(0)}{\boldsymbol{v}}, ..., \overset{(n-1)}{\boldsymbol{u}}, \overset{(n-1)}{\boldsymbol{v}}), \\ \mu h^2 \nabla_\alpha \nabla^\alpha \overset{(n)}{v}_3 + 3 \left[\lambda h \overset{(n)}{\theta} + (\lambda + 2\mu) \overset{(n)}{v} \right] = \overset{(n)}{B}_3 (\overset{0}{\boldsymbol{u}}, \overset{(0)}{\boldsymbol{v}}, ..., \overset{(n-1)}{\boldsymbol{u}}, \overset{(n-1)}{\boldsymbol{v}}), \end{cases}$$

where $\overset{(n)}{\theta} = \nabla_\alpha \overset{(n)}{u}{}^\alpha$, $\overset{(n)}{\rho} = \nabla_\alpha \overset{(n)}{v}{}^\alpha$, $\mathbf{U} = \mathbf{u} + p_1 \left(\frac{x_3}{h} \right) \mathbf{v}$.

The general solution of the system is expressed by six poliharmonic functions on the midsurface S.

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