Reports of Enlarged Session of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 20, №3, 2005

ON ONE VARIATIONAL INTERPRETATION OF BITSADZE-SAMARSKII NON-LOCAL BOUNDARY VALUE PROBLEM FOR THE BIHARMONIC EQUATION

Lobjanidze G.

Iv. Javakhishvili Tbilisi State University

In this article one variational concept of the Bitsadze-Samarskii nonlocal boundary value problem for the biharmonic equation in the rectangle is considered [1].

Problem. Find the function $u(x,y) \in C^1(\overline{G}) \cap C^4(G)$ satisfying the conditions

$$\begin{aligned} &\Delta^2 u(x,y) = f(x,y), \quad \text{for} \quad (\mathrm{x,y}) \in \mathrm{G}, \\ &u(x,y)|_{\Gamma} = 0, \qquad \frac{\partial u(x,y)}{\partial \nu}\Big|_{\partial G} = 0, \\ &u(x,y)|_{\Gamma_{-\xi}} = u(x,y)|_{\Gamma_0}, \end{aligned} \tag{1}$$

where $G = \{(x,y)| \ -a < x < 0, \ 0 < y < b\}$ is the rectangle, a and b are the given positive constants, $f(x,y)$ is the given function $f(x,y) \in C(\overline{G})$, ν is the external unitary normal of the boundary of G, Γ_t is the intersection of the line $x = t$ with the set $\overline{G} = G \cup \partial G$, $\Gamma = \partial G \backslash \Gamma_0$, $\xi \in]-a, 0[$ is the given number.

Let us denote by $D(\overline{G})$ the lineal of all the real functions $u(x,y)$ satisfying the following conditions:

1. $u(x,y)$ is defined almost everywhere on $\overline{G} \backslash \Gamma_0$, and the boundary value $u(0,y)$ is defined almost everywhere on Γ_0.

2. $u(x,y) \in L_2(G)$, $u(0,y) \in L_2(0,b)$.

We note that the definition of the function $u(x,y) \in D(G)$ means the definition of the couple $u(x,y), u(0,y))$, $(x,y) \in \overline{G} - \Gamma_0, y \in [0,b]$.

The same definition of the function $u(x,y)$ of the lineal $D(\overline{G})$ and for the function $u(x,y) \in L_2(G)$ defined almost everywhere on $\overline{G} \backslash \Gamma_0$ will be clear from the context.

Two functions $u_1(x,y)$ and $u_2(x,y)$ are assumed as the same element of $D(\bar{G})$ if $u_1(x,y) = u_2(x,y)$ almost everywhere on $\overline{G} \backslash \Gamma_0$ and $u_1(0,y) = u_2(0,y)$ almost everywhere on $[0,b]$.

Let us define on $D(\overline{G})$ the operator of symmetrical extension $\mathcal{T}$ as follows

$$\mathcal{T}\, u(x,y) = \begin{cases} u(x,y), & \text{if} \quad (\mathrm{x,y}) \in \overline{\mathrm{G}}, \\ -u(-x,y) + 2u(0,y), & \text{if} \quad (\mathrm{x,y}) \in \overline{\mathrm{Q}}. \end{cases}$$

where $Q = \{(x,y)| \ 0 < x < \xi, \ 0 < y < b\}$.

Let us note that the operator $\mathcal{T}$ associates to every function $u(x,y)$ of the lineal $D(\overline{G})$ the function $\widetilde{u}(x,y) = \mathcal{T}\, u(x,y)$ defined almost everywhere on $\overline{G} \cup \Gamma_0$ in such a way that the function $\widetilde{u}(x,y) - u(0,y)$ is the odd function with respect to the variable x almost everywhere on $[-\xi, \xi]$ for the almost all $y \in [0,b]$.

For two arbitrary functions $u(x,y)$ and $v(x,y)$ from the linear $D(\overline{G})$ we define the scalar product

$$[u,v]=\int\limits_0^b\int\limits_{-\xi}^{\xi}\int\limits_{-a}^{x}\widetilde{u}(s,y)\,\widetilde{v}(s,y)\,ds\,dx\,dy. \tag{2}$$

After the introduction of the scalar product (2) the lineal $D(\overline{G})$ becomes the pre-Hilbert space, which we denote by $H(\overline{G})$. The norm originated from the scalar product (2) in $H(\overline{G})$ we denote by $\|\cdot\|_H$:

$$\|u\|_H^2=\int\limits_0^b\int\limits_{-\xi}^{\xi}\int\limits_{-a}^{x}\widetilde{u}(s,y)^{\,2}\,ds\,dx\,dy. \tag{3}$$

Theorem 1. *The norm defined on the linear $D(\overline{G})$ by the formula*

$$\|u\|^2=\|u(x,y)\|^2_{L_2(G)}+\|u(0,y)\|^2_{L_2(o,b)}$$

is equivalent to the norm $\|\cdot\|_H$.

Consequence. *$H(\overline{G})$ is the Hilbert space.*

Let, the area of definition of the operator $A=\Delta^2$ is the lineal $D_A(\overline{G})$ of the functions from the space $H(\overline{G})$ for the elements $u(x,y)$ of which the following conditions are fulfilled:

1. $u(x,y)\in C^4(\overline{G})$, $\quad \dfrac{\partial^k u(0,y)}{\partial x^k}=0 \quad \forall\, y\in[0,b], \quad k=2,4\,;$
2. $u(x,y)|_\Gamma=0$, $\quad \dfrac{\partial u}{\partial \nu}\Big|_{\partial G}=0$, $\quad \dfrac{\partial^2 u}{\partial x^2}(-\xi,y)=0, \quad u(-\xi,y)=u(0,y), \quad \forall\, y\in[0,b].$

Theorem 2. *The lineal $D_A(G)$ is danced in the space $H(\overline{G})$.*

Hence, the operator A acts from the lineal $D_A(\overline{G})$ danced in the space $H(\overline{G})$ to the space $H(\overline{G})$.

Lemma 1. *For an arbitrary function $u(x,y)$ of the lineal $D_A(\overline{G})$ the following identity is valid*

$$\mathcal{T}Au=A\mathcal{T}u.$$

Lemma 2. *For two arbitrary functions $u(x,y)$ and $v(x,y)$ of the lineal $D_A(\overline{G})$ we have*

$$\int\limits_{-\xi}^{\xi}\frac{\partial\widetilde{u}(x,y)}{\partial x}\,\widetilde{v}(x,y)\,dx=0 \quad \forall\, y\in[0,b].$$

Taking into account Lemmas 1,2 and Poincare's and Fridrikh's inequalities [3] we easily obtain the following

Theorem 3. *Operator $A=\Delta^2$ is positively defined on the lineal $D_A(\overline{G})$.*

Let us note, that in the article [2] by using the scalar product

$$(u,v)=\int\limits_0^b\int\limits_{-\xi}^{0}\int\limits_{-a}^{x}u(s,y)\,v(s,y)\,ds\,dx\,dy \tag{4}$$

was proved the inequality of the positively definition (the uniqueness of the solution) for the corresponding operator of various non-local boundary value problems.

In the articles [4,5] by using of the scalar product of the type (2) was proved that the operators corresponding to the simple non-local boundary value problems are positively defined on the corresponding lineals.

Hence the operator $A = \Delta^2$ is positively defined on the lineal $D_A(\overline{G})$ in the Hilbert space $H(\overline{G})$. So we can follow the standard way.

After introducing the scalar product on $[u,v]_A$ the lineal $D_A(\overline{G})$ becomes the pre-Hilbert space which we denote by $S_A(\overline{G})$. By $H_A(\overline{G})$ we denote the Hilbert space obtained after completion of $S_A(\overline{G})$ by the norm $\|\cdot\|_A$. For every function $f(x,y) \in H(\overline{G})$ the quadratic function

$$F(u) = [u,u]_A - 2\,[f,u] \tag{5}$$

has the unique function $u_0(x,y) \in H_A(\overline{G})$, which minimizes the functional (3) and the function $u_0(x,y)$ satisfies the condition $[u_0,u]_A = [f,u]$ for every $u(x,y) \in D_A(\overline{G})$. When the function $f(x,y)$ belongs to the lineal $R_A(\overline{G})$ of the operator A, then the function $u_0(x,y) \in D_A(\overline{G})$ is the solution of the following problem

$$\begin{aligned} &\Delta^2 u(x,y) = f(x,y), \quad \text{for} \quad \mathrm{(x,y) \in \overline{G}}, \\ &u(x,y)|_\Gamma = 0, \quad \frac{\partial u(x,y)}{\partial \nu}\Big|_{\partial G} = 0, \\ &u(x,y)|_{\Gamma_{-\xi}} = u(x,y)|_{\Gamma_0}. \end{aligned} \tag{6}$$

It is naturally to introduce the following definition

Definition. The function $u_0(x,y)$ which minimizes the quadratic functional (3) in the space $H_A(\overline{G})$ is called the generalized solution of the equation (4).

Remark. For the Problem (1) $f(x,y) \in C(\overline{G})$ and the general solution originated from the couple $(f(x,y), f(0,y))$ corresponding to the function $f(x,y) \in H(\overline{G})$ is not generally equal to the Bitsadze-Samarski solution of the Problem (1), for example it is equal in the case $f(x,y) \in R_A(\overline{G})$. The problem of finding the function $f_0(x,y)$ for which the general solution of the problem (1) corresponding to the couple $(f(x,y), f(0,y))$ is equal to the ordinary solution is to be studied. Two following theorems are characterized the space $H_A(\overline{G})$.

Theorem 4. *The function $u(x,y) \in H(\overline{G})$ belongs to the space $H_A(\overline{G})$ if and only if the following relations are fulfilled*

$$u(x,y) \in W_2^2(G)$$

and the traces $u(x,y)|_{\Gamma_{-\xi}}$, $u(x,y)|_{\Gamma_0}$ and the boundary value $u(0,y)$ are the same elements of the space $W_2^2(0,b)$

$$u(x,y)|_\Gamma = 0, \quad \frac{\partial u(x,y)}{\partial \nu}|_\Gamma = 0.$$

Theorem 5. *The norm*

$$|||u|||^2 = \|u(x,y)\|^2_{W_2^2(G)} + \|u(0,y)\|^2_{W_2^2(0,b)}$$

defined in the space $H_A(\overline{G})$ is equivalent to the norm $\| \cdot \|_A$.

REFERENCES

1. Bitsadze A.V., Samarskii A.A. On some simplified generalization of the linear elliptic problems, Dokl. AN SSSR, 1969, vol. 185, N 4, pp. 739–740 (in Russian).

2. Gordeziani D.G. On the methods of solution for one class of non-local boundary value problems, University Press, Tbilisi, 1981. p. 32 (in Russian).

3. Rektorys K. Variational Methods in Mathematics,Science and Engineering, Prague, 1980.

4. Lobjanidze G. Remark on the variational formulation of Bitadze-Samarski nonlocal problem. Rep. of Enlanged Session of the seminar of VIAM vol. 16, N 3, 2001 pp. 102-103.

5. Lobjanidze G. On the variational formulation of Bitadze-Samarski equation $\Delta u - \lambda u(x, y) = f$, Rep. of Enlanged Session of the seminar of VIAM vol. 16, N 3, 2001 pp. 102-103.

Received 25. IV. 2005; accepted 20. VI. 2005.