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FINITE DIFFERENCE SCHEMES FOR MULTIDIMENSIONAL PARABOLIC TYPE EQUATION WITH CONSTANT COEFFICIENTS

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Abstract

The paper deals with a simple unified algorithm for construction of absolutely stable economical schemes to solve multidimensional parabolic type equations, where each difference equation completely approximates the given differential equation. It is wortly to note that for the first time the constructed schemes are dependent on the dimension p only (they are not dependent on the weight).

1. Introduction. The finite difference schemes which are now known as "alternative direction sweep" method was suggested first in 1955 simultaneously by Peaceman, Rachford and Douglas in [1.2]. These papers have become a basis to develop absolutely stable schemes (the so called economical schemes). Afterwards the schemes have been extended and deepened by Douqlas, Rachford, Baker, Oliphant, Brian, Samarshi, Yanenko, Marchuk, Gordeziani and others. We refer the reader to papers [3-5] and the bibliography therein.

The proposed method allows to write down new schemes for an arbitrary dimension p. In particular, when $p = 1$, $p = 2$ and the right-hand side of the equation vanishes, we get respectively the Krank-Nikolski and Douglas-Rachford schemes (see [1-5]).

For simplicity consider the case of constant coefficients. Although with minor changes everything below can be stated for the case of a general second order equation with non-constant coefficients.

2. Setting of the problem, variation problem and difference schemes. Consider the first initial-boundary problem for a p-dimensional heat conductivity equation, where

$$\frac{\partial u}{\partial t} = Lu + f, \quad Lu = \sum_{i=1}^{p} L_i u, \quad L_i u = \frac{\partial^2 u}{\partial x_i^2}, \quad x \in G, \quad t \in (0, T] \tag{1}$$

$$u|_\Gamma = 0, \quad u(x, 0) = u_0(x). \tag{2}$$

Let $G = G_{op}$ be p-dimensional cube $0 \le x_i \le 1$, $i = \overline{1, p}$, $\overline{\omega}_h = \{(i_1 h_1, \ldots, i_p h_p) \in G\}$ be the cube type net with the step h with respect to the variable x_i, $h = \dfrac{1}{N_1}$, $\overline{\omega}_\tau$ -be the net with the step $\tau = \dfrac{T}{N_0}$ on the interval $0 \le t \le T$.

Let us consider first the algorithm of the construction of schemes for $p = 1$. After established the rule we can write it in general case of arbitrary p.

To this end we rewrite equation (1) in the form

$$\beta\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial t^2}=\left(\frac{\partial^2 u}{\partial t^2}+\beta\frac{\partial^2 u}{\partial x_1^2}\right)+\beta f,$$

where β is a so far, not yet defined nonzero real number. In the sequel, both the stability and approximation of finite difference schemes will be dependent just on the selection of β and weight α.

In order to define $u(x_1,t)$, instead of problems (1) - (2), let us apply the well-known Hamilton principle:

$$I(u)=\int_{t_0}^{t_n}\left\{\int_0^1\left[\beta u\frac{\partial u}{\partial t}+\left(\frac{\partial u}{\partial t}\right)^2+\frac{1}{2n_1}\sum_{i=1}^{2n_1}\left[A_1^{(i)}\left(\frac{\partial u}{\partial \ell_i}\right)^2+B_1^{(i)}\left(\frac{\partial u}{\partial \ell_{i+1}}\right)^2+\right.\right.\right.$$
$$\left.\left.\left.+C_1^{(i)}\frac{\partial u}{\partial \ell_i}\frac{\partial u}{\partial \ell_{i+1}}\right]-2\beta fu\right]dx_1\right\}dt\to min,$$

i.e. if the function $u(x_1,t)$ is the solution to the given problem, then it provides the minimal value for the functional $I(u)$ in any time interval t_n-t_0 $(t_0=0)$,

$$\text{where } A_1^{(i)}=\frac{\cos^2\alpha_{i+1}+\beta\sin^2\alpha_{i+1}}{\sin^2(\alpha_{i+1}-\alpha_i)},\ B_1^{(i)}=\frac{\cos^2\alpha_i+\beta\sin^2\alpha_i}{\sin^2(\alpha_{i+1}-\alpha_i)},$$

$$C_1^{(i)}=\frac{-2(\cos\alpha_i\cos\alpha_{i+1}+\beta\sin\alpha_i\sin\alpha_{i+1})}{+}\sin^2(\alpha_{i+1}-\alpha_i).$$

The direction $\vec{\ell_i}$ are defined by means of the angles α_i as follows :

$$\vec{\ell}=(\ell_1,\ell_2,\dots,\ell_{2n_1})\Rightarrow(\alpha_1,\alpha_2,\dots,\alpha_{2n_1}),\quad \alpha_{n_1+i}=\pi+\alpha_i,\quad \vec{\ell_i}=-\vec{\ell}_{n_1+i},\quad i=\overline{1,n_1}$$

Let us consider one of the above possibilities. Namely when in the elementary cell containing a basis knot (x_i,t_j) (it is a rectangle with the centroid (x_i,t_j) the four directions ℓ_i, $i=\overline{1,4}$ $(n_1=2)$ get out from the point (x_i,t_j), where

$$\alpha_1=\pi-\alpha_2,\quad \alpha_3-\alpha_2=2\alpha_1,\quad \alpha_4-\alpha_3=\pi-2\alpha_1,\quad \alpha_1'-\alpha_4=2\alpha_1,\quad \alpha_1'=2\pi+\alpha_1.$$

Here α_1 is the counterclockwise angle between the axis ox_1 and the direction ℓ_1.

Let us exploit the mean value formula and write down the difference functional (with respect to basis knot (x_i,t_j) and to the fictitious knots $\left(x_i+\frac{h}{2},t_j+\frac{\tau}{2}\right)$, $\left(x_i-\frac{h}{2},t_j+\frac{\tau}{2}\right)$, $\left(x_i-\frac{h}{2},t_j-\frac{\tau}{2}\right)$, $\left(x_i+\frac{h}{2},t_j-\frac{\tau}{2}\right)$, where τ is the net step along the T axis, $t_j=j\tau$, $j=0,1,...$; while h is the step along the ox_1 axis) corresponding to the functional . We call the knots to the fictitius, since the values of a function in the finite difference schemes are not involved because they are cancelled.

In the sequel we leave the same notation U for the net function

$$I_h(u)=\left\{\left[\frac{\beta}{2}u(u_t+u_{\bar t})+(u_t^2+u_{\bar t}^2)\right]_{(x_i,t_j)}+\frac{1}{4}\sum_{i=1}^{4}\left[A_1^{(i)}u_{\ell_i}^2+B_1^{(i)}u_{\ell_{i+1}}^2+\right.\right.$$

$$+C_1^{(i)}u_{\ell_i}u_{\ell_{i+1}}\Big]_{(x_i,t_j)}+\frac{1}{4}\left(\sum_{i=1}^{4}C_1^{(i)}u_{\ell_i}u_{\ell_{i+1}}\right)_{\left(x_i+\frac{h}{2},t_j+\frac{\tau}{2}\right),\cdots,\left(x_i+\frac{h}{2},t_j-\frac{\tau}{2}\right)}\Bigg\}S_0,$$

where $S_0=4h\tau$, is the area of elementary cell. We use the following notations :

$$(u_{\ell_1})_{(x_i,t_j)}=\frac{u(x_i+h,t_j+\tau)-(x_i,t_j)}{\sqrt{h^2+\tau^2}},\ (u_{\bar{\ell}_1})_{(x_i,t_j)}=\frac{u(x_i,t_j)-u(x_i-h,t_j-\tau)}{\sqrt{h^2+\tau^2}}$$

$$u_{x_i}=\frac{u(x_i+h,t_j)-u(x_i,t_j)}{h},\ u_{\bar{x}_i}=\frac{u(x_i,t_j)-u(x_i-h,t_j)}{h},$$

$$u_{\overset{0}{x}}=\frac{u_x+u_{\overrightarrow{x}}}{2h},\ u_t=\frac{u(x_i,t_j+\tau)-u(x_i,t_j)}{\tau}=\frac{\hat{u}-u}{\tau},$$

$$u_{\bar{t}}=\frac{u(x_i,t_j)-u(x_i,t_j-\tau)}{\tau}=\frac{u-\check{u}}{\tau},\quad u_{\overset{0}{x}}=\frac{\hat{u}-\check{u}}{2\tau}.$$

If we insert the values of difference operators in the obtained functional, use the Hamilton principle and carry out elementary but routine work, then we obtain the following difference scheme with respect to the basic knot point (x_i,t_j)

$$u_{\overset{0}{t}}+\frac{h^2+\beta\tau^2}{2\beta h^2}u_{t\bar{t}}=\frac{(h^2+\tau^2)(h^2+\beta\tau^2)}{\beta\cdot(2\tau h)^2}(u_{\ell_1\bar{\ell}_1}+u_{\ell_2\bar{\ell}_2})-\frac{h^2-\beta\tau^2}{2\beta\tau^2}u_{x_1\bar{x}_1}+f_{ij}.$$

After a few transformations we can rewrite it in the following canonical form

$$u_{\overset{0}{t}}+\tau^2Ru_{t\bar{t}}=u_{x_1\bar{x}_1}+f,\quad (f=f(x_i,t_j)=f_{ij}), \tag{3}$$

where

$$R=\frac{h^2+\beta\tau^2}{4\beta\tau^2}L,\ L=-\Delta_{11},\ \Delta_{11}u=u_{x_1\bar{x}_1}.$$

If in scheme (3) we replace β by $\beta=\dfrac{1}{\sigma^*}\dfrac{h^2}{\tau^2}$, then we get the equivalent to (3) scheme, which we call σ-parametric basis scheme

$$u_{\overset{0}{t}}+\sigma\tau^2Ru_{t\bar{t}}=u_{x_1\bar{x}_1}+f, \tag{4}$$

where $\sigma=\dfrac{1+\sigma^*}{4}>0,\ \left(\sigma\neq\dfrac{1}{4}\right).$

If we repeat word for word the method of receiving the finite difference schemes, we obtain the following schemes

$$u_{\overset{0}{t}}+\sigma\tau^2R_ku_{t\bar{t}}=\sum_{i=1}^{p}u_{x_i\bar{x}_i}+f,\ k=\overline{1,p}. \tag{5}$$

Note that (5) represents p independent schemes. The fixed k determines the variable x_k with respect to which the "sweep" method is being used for finding solution U.

3. Weight schemes and study of the difference schemes. Having the rule described, we can write down the α-weight schemes by means of scheme (5) for a general p $\left(\text{where } \sigma = \dfrac{1-\alpha}{2}\right)$

$$\frac{u^{n+k}-u^{n+k-2}}{2\tau} - \frac{1-\alpha}{2}\Delta_{kk}(u^{n+k}+u^{n+k-2}) = \alpha\Delta_{kk}u^{n+k} + \sum_{\substack{i=0\\ i\neq k}}^{p}\Delta_{ii}u^{n+k-2} + f^{n+k-1},\ k=\overline{1,p}. \tag{6}$$

In the sequel, we exploit the following representation of scheme (6) :

$$\frac{u^{n+k}-u^{n+k-2}}{2\tau} = \Delta_{kk}\left(\frac{p}{2}u^{n+k} + \left(1-\frac{p}{2}\right)u^{n+k-2}\right) + \sum_{\substack{i=0\\ i\neq k}}^{p}\Delta_{ii}u^{n+k-2} + f^{n+k-1}, \quad k=\overline{1,p} \tag{7}$$

or well the fractional step notations

$$\frac{u^{n+\frac{k}{p}}-u^{n+\frac{k-1}{p}}}{\frac{1}{p}\tau} = \Delta_{kk}\left(\frac{p}{2}u^{n+\frac{k}{p}} + \left(1-\frac{p}{2}\right)u^{n+\frac{k-1}{p}}\right) + \sum_{\substack{i=0\\ i\neq k}}^{p}\Delta_{ii}u^{n+\frac{k-1}{p}} + f^{n+\frac{2k-1}{2p}}, \tag{8}$$

$$k=\overline{1,p},\ (\text{where } \alpha = p-1).$$

Let us prove the unconditional stability of the scheme (6). Suppose

$$u^n = \rho^n e^{i(k_1x_1+k_2x_2+\cdots+k_px_p)},\ \rho^n = e^{\omega nt},\ (i=\sqrt{-1}). \tag{9}$$

Assuming that $f=0$ and substituting (9) into equation (6), we obtain the following dispersion equation First, let us write down the dispersion equation for any p:

$$\left(1+\frac{1+\alpha}{2}a_i\right)\rho^2 + \left(\frac{1+\alpha}{2}a_i + \sum_{\substack{i=0\\ i\neq k}}^{p}a_i - 1\right) = 0,\ i=\overline{1,p}$$

where, in general $a_j = 4r_i\sin^2\dfrac{k_jh_j}{2}$, $r_j = \dfrac{2\tau}{h_j^2}, j=\overline{1,p}$.

For the stability it is necessary that

$$|\rho| = |\rho_1\cdot\rho_2\cdots\rho_p| =$$

$$\sqrt{\frac{\left|1-\frac{1-\alpha}{2}a_1 - a_2 - \cdots - a_p\right|}{1+\frac{1+\alpha}{2}a_1}\cdots\frac{\left|1-\frac{1-\alpha}{2}a_p - a_1 - a_2 - \cdots - a_{p-1}\right|}{1+\frac{1+\alpha}{2}a_p}} \le 1.$$

Asymptotically, i.e. when the value of $\dfrac{\tau}{h_i^2}$ is large enough, we get the following inequality for $\rho(\alpha)$

$$|\rho(p-1)| = \sqrt{\left(\frac{\left|1-\frac{2p-\alpha-1}{2}a\right|}{1+\frac{1+\alpha}{2}a}\right)^p} = \sqrt{\left(\frac{\left|1-\frac{p}{2}a\right|}{1+\frac{p}{2}a}\right)^p} \le 1$$

when $\alpha = p - 1$, we get unconditional condition of stability

$$0 < \frac{\tau}{h^2} < \infty \quad p = 1, 2, \dots .$$

This permits us to make the following conclusion : the two-layer finite difference schemes (7), or which is the same, the schemes (8) which have the exactness $O(\tau^2 + |h|^2)$, represent a generalization of existing economic schemes for multidimensional parabolic equations with constant coefficients. Besides, the choice with weight α permited the schemes to be dependent only on the dimension p.

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