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PARTIALLY CLAMPED SHALLOW SHELLS COMPOSITION OF BINARY MIXTURE

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In this paper a version of linear theory for a body composed of two isotropic materials suggested by Green-Naghdi-Steel [1], [2], [3] is studied. For this types of shallow shells is consider following three-dimensional problem: the stresses is given on the upper and lower faces and on the portion of lateral surface, and remainder part of boundary is clamped. By asymptotic analysis method (Lions, Ciarlet, Destuynder [4], [5]) is obtained and studed corresponding two-dimensional problem. It is shown, that the limiting mean of particular flexures of two components of the mixture are equal one another, when $\varepsilon \to 0$ (the semithickness of shell ε be a small parameter). In case of plate will obtain analogous equation of classical bending biharmonic equation with corresponding boundary condition [6].

We assume that an origin and orthonormal basis $\{\mathbf{e}_i\}$ have been chosed in the three-dimensional Euclidean space. Let ω be a connected domain with Lipschitz boundary in the plane spanned by the vectors $\mathbf{e}_\alpha$ (Latin and Greek lower indexes take value accordingly 1,2,3 and 1,2. Under repeating indexes we mean sumation). For each $\varepsilon > 0$, let

$$\Omega^\varepsilon := \omega \times]-\varepsilon, \varepsilon[, \quad \Gamma_+^\varepsilon := \omega \times \{\varepsilon\}, \quad \Gamma_-^\varepsilon := \omega \times \{-\varepsilon\},$$

$x^\varepsilon := (x_i^\varepsilon) = (x_1, x_2, x_3^\varepsilon)$ is any point of the set $\overline{\Omega^\varepsilon}$,

$$\partial_\alpha = \partial_\alpha^\varepsilon := \frac{\partial}{\partial x_\alpha} \quad \text{and} \quad \partial_3^\varepsilon := \frac{\partial}{\partial \mathrm{x}_3^\varepsilon}.$$

The middle surface of shell will define following

$$\widehat{\omega}^\varepsilon := \left\{(x_1, x_2, \theta^\varepsilon(x_1, x_2)) \in \mathbb{R}^3; \ (x_1, x_2) \in \omega\right\},$$

where θ^ε is $C^2(\overline{\omega})$ class function for all positive ε.

For each $\varepsilon > 0$ the mapping $\mathbf{\Theta}^\varepsilon := (\Theta_i^\varepsilon): \ \overline{\Omega}^\varepsilon \to \mathbb{R}^3$, will define following form $\mathbf{\Theta}^\varepsilon(x^\varepsilon) = (x_1, x_2, \theta^\varepsilon(x_1, x_2)) + x_3^\varepsilon \mathbf{a}_3^\varepsilon(x_1, x_2)$ for all $x^\varepsilon \in \overline{\Omega}^\varepsilon$, where

$$\mathbf{a}_3^\varepsilon := \{\alpha^\varepsilon\}^{\frac{-1}{2}} (-\partial_1\theta^\varepsilon, \ -\partial_2\theta^\varepsilon, \ 1), \quad \alpha^\varepsilon := |\partial_1\theta^\varepsilon|^2 + |\partial_2\theta^\varepsilon|^2 + 1.$$

It mean that for each $\varepsilon > 0$ the mapping $\mathbf{\Theta}^\varepsilon: \ \overline{\Omega}^\varepsilon \to \mathbf{\Theta}^\varepsilon(\overline{\Omega}^\varepsilon)$ is a C^1-diffeomorphism. If $\widehat{\Omega}^\varepsilon := \mathbf{\Theta}^\varepsilon(\Omega^\varepsilon)$, then $\left\{\widehat{\Omega}^\varepsilon\right\}^- = \mathbf{\Theta}^\varepsilon(\overline{\Omega}^\varepsilon)$. $\widehat{x}^\varepsilon = (\widehat{x}_i^\varepsilon)$ is denote any point of $\left\{\widehat{\Omega}^\varepsilon\right\}^-$. $\widehat{\partial}_i^\varepsilon := \dfrac{\partial}{\partial \widehat{x}_i^\varepsilon}$.

Let $\gamma_0 \subset \partial\omega$, $length \gamma_0 > 0$; $\widehat{\Gamma}_0^\varepsilon := \mathbf{\Theta}^\varepsilon(\Gamma_0^\varepsilon)$, $\Gamma_0^\varepsilon := \gamma_0 \times [-\varepsilon, \varepsilon]$; $\widehat{\Gamma}_+^\varepsilon := \mathbf{\Theta}^\varepsilon(\Gamma_+^\varepsilon)$, $\widehat{\Gamma}_-^\varepsilon := \mathbf{\Theta}^\varepsilon(\Gamma_-^\varepsilon)$.

Let the domain $\widehat{\Omega}^\varepsilon$ is composition of isotropic material of binnary mixture. Then variational formulation considered problem have form $P(\widehat{\Omega}^\varepsilon)$*problem*: Let $\widehat{F}_i^\varepsilon \in (L^2(\widehat{\Omega}^\varepsilon))^2$, $\widehat{g}_i^\varepsilon = (\widehat{g}_i'^\varepsilon,\ \widehat{g}_i''^\varepsilon)^T \in (L^2(\widehat{\Gamma}_+^\varepsilon \cup \widehat{\Gamma}_-^\varepsilon))^2$. Find

$$\widehat{\mathbf{u}}^\varepsilon \in \mathbf{V}(\widehat{\Omega}^\varepsilon) := \left\{ \widehat{\mathbf{v}}^\varepsilon = (\widehat{v}_1'^\varepsilon, \widehat{v}_2'^\varepsilon, \widehat{v}_3'^\varepsilon, \widehat{v}_1''^\varepsilon, \widehat{v}_2''^\varepsilon, \widehat{v}_3''^\varepsilon) \in (H^1(\widehat{\Omega}^\varepsilon))^6,\ \ \widehat{\mathbf{v}}^\varepsilon = 0 \ \text{ on } \ \widehat{\Gamma}_0^\varepsilon \right\},$$

such that satisfies the variational equations

$$\int\limits_{\widehat{\Omega}^\varepsilon} \left\{ \left(\Lambda^\varepsilon \widehat{e}_{pp}^\varepsilon(\widehat{\mathbf{u}}^\varepsilon)\right)^T \widehat{e}_{qq}^\varepsilon(\widehat{\mathbf{v}}^\varepsilon) + 2\left(M^\varepsilon \widehat{e}_{ij}^\varepsilon(\widehat{\mathbf{u}}^\varepsilon)\right)^T \widehat{e}_{ij}^\varepsilon(\widehat{\mathbf{v}}^\varepsilon) - 2\lambda_4^\varepsilon \widehat{h}_{ij}^\varepsilon(\widehat{\mathbf{u}}^\varepsilon)\widehat{h}_{ij}^\varepsilon(\widehat{\mathbf{v}}^\varepsilon) \right\} d\widehat{x}^\varepsilon$$

$$= \int\limits_{\widehat{\Omega}^\varepsilon} (\widehat{F}_i^\varepsilon)^T \widehat{v}_i^\varepsilon d\widehat{x}^\varepsilon + \int\limits_{\widehat{\Gamma}_+^\varepsilon \cup \widehat{\Gamma}_-^\varepsilon} (\widehat{g}_i^\varepsilon)^T \widehat{v}_i^\varepsilon d\widehat{\Gamma}^\varepsilon \qquad \text{forall } \ \widehat{\mathbf{v}}^\varepsilon \in \mathbf{V}(\widehat{\Omega}^\varepsilon).$$

Here $\widehat{v}_j^\varepsilon = (\widehat{v}_j'^\varepsilon, \widehat{v}_j''^\varepsilon)^T$ is a column matrix of the components of partial displacement; $\widehat{e}_{ij}^\varepsilon = \left(\widehat{e}_j'^\varepsilon, \widehat{e}_{ij}''^\varepsilon\right)^T = \dfrac{1}{2}\left(\widehat{\partial}_i^\varepsilon \widehat{u}_j^\varepsilon + \widehat{\partial}_j^\varepsilon \widehat{u}_i^\varepsilon\right)$ is a column matrix composite the components of the strain tensor; $\widehat{h}_{ij}^\varepsilon(\widehat{\mathbf{v}}^\varepsilon) = \dfrac{1}{2}\left(\widehat{\partial}_i^\varepsilon \widehat{v}_j'^\varepsilon - \widehat{\partial}_j^\varepsilon \widehat{v}_i'^\varepsilon + \widehat{\partial}_j^\varepsilon \widehat{v}_i''^\varepsilon - \widehat{\partial}_i^\varepsilon \widehat{v}_j''^\varepsilon\right)$ - components of so called rotation tensor;

$$\Lambda^\varepsilon := \begin{pmatrix} \lambda_1^\varepsilon & \lambda_3^\varepsilon \\ \lambda_3^\varepsilon & \lambda_2^\varepsilon \end{pmatrix}, \quad M^\varepsilon := \begin{pmatrix} \mu_1^\varepsilon & \mu_3^\varepsilon \\ \mu_3^\varepsilon & \mu_2^\varepsilon \end{pmatrix},$$

$\lambda_1^\varepsilon,\ \lambda_2^\varepsilon,\ \lambda_3^\varepsilon,\ \lambda_4^\varepsilon, \mu_1^\varepsilon, \mu_2^\varepsilon, \mu_3^\varepsilon$ - elasticity modulus.

Theorem 1. *Let satisfies following condition*

$$\lambda_4^\varepsilon \le 0, \quad \lambda_1^\varepsilon + \frac{2}{3}\mu_1^\varepsilon > 0, \quad det M^\varepsilon > 0, \quad det\left(\Lambda^\varepsilon + \frac{2}{3}M^\varepsilon\right) > 0,$$

then for each $\varepsilon > 0$, problem $P(\widehat{\Omega}^\varepsilon)$ have one and only one solution.

The problem $P(\widehat{\Omega}^\varepsilon)$ be transformate into a equivalent problem posed over a domain independent of ε.

$$\Omega := \omega \times]-1, 1[,\ \ \Gamma_\pm := \omega \times \{\pm 1\},\ \Gamma_0 := \gamma_0 \times [-1, 1].$$

$$\pi^\varepsilon : x = (x_i) \in \overline{\Omega} \to x^\varepsilon = (x_i^\varepsilon) = (x_1, x_2, \varepsilon x_3) \in \overline{\Omega}^\varepsilon.$$

The functions $\widehat{\mathbf{u}}^\varepsilon$, $\widehat{\mathbf{v}}^\varepsilon \in \mathbf{V}(\widehat{\Omega}^\varepsilon)$ associate with the scalar field $\mathbf{u}(\varepsilon)$ and the scalar functions following form

$$\widehat{u}_\alpha^\varepsilon(\widehat{x}^\varepsilon) = \varepsilon^2 u_\alpha(\varepsilon)(x) \ \text{ and } \ \widehat{\mathrm{v}}_\alpha^\varepsilon(\widehat{\mathrm{x}}^\varepsilon) = \varepsilon^2 \mathrm{v}_\alpha(\mathrm{x});$$

$\widehat{u}_3^\varepsilon(\widehat{x}^\varepsilon) = \varepsilon u_3(\varepsilon)(x)$ and $\widehat{\mathrm{v}}_3^\varepsilon(\widehat{\mathrm{x}}^\varepsilon) = \varepsilon \mathrm{v}_3(\mathrm{x})$; for each $\widehat{x}^\varepsilon = \mathbf{\Theta}^\varepsilon(\pi^\varepsilon(x)) \in \left\{\widehat{\Omega}^\varepsilon\right\}^-$.

Let there exists the constants $\lambda_j,\ \lambda_4,\ \mu_j$ and the functions $g_i \in (L^2(\Gamma_+ \cup \Gamma_-))^2,\ \ F_i \in (L^2(\Omega))^2$, independent of ε such that

$$\lambda_j^\varepsilon = \lambda_j,\ \ \lambda_4^\varepsilon = \lambda_4,\ \mu_j^\varepsilon = \mu_j,\ \widehat{F}_\alpha^\varepsilon(\widehat{x}^\varepsilon) = \varepsilon^2 F_\alpha(x),\ \widehat{F}_3^\varepsilon(\widehat{x}^\varepsilon) = \varepsilon^3 F_3(x)$$

$$\text{forall}\ \ \widehat{\mathrm{x}}^\varepsilon = \boldsymbol{\Theta}^\varepsilon(\pi^\varepsilon \mathrm{x}) \in \widehat{\Omega}^\varepsilon,$$

$$\widehat{g}_\alpha^\varepsilon(\widehat{x}^\varepsilon) = \varepsilon^3 g_\alpha(x)\ \text{ and }\ \widehat{\mathrm{g}}_3^\varepsilon(\widehat{\mathrm{x}}^\varepsilon) = \varepsilon^4 \mathrm{g}_3(\mathrm{x})\ \text{ forall }\ \widehat{\mathrm{x}}^\varepsilon \in \boldsymbol{\Theta}^\varepsilon(\pi^\varepsilon \mathrm{x}) \in \widehat{\Gamma}_+^\varepsilon \cup \widehat{\Gamma}_-^\varepsilon.$$

$$\Lambda := \begin{pmatrix} \lambda_1 & \lambda_3 \\ \lambda_3 & \lambda_2 \end{pmatrix}, \quad M := \begin{pmatrix} \mu_1 & \mu_3 \\ \mu_3 & \mu_2 \end{pmatrix}.$$

The function θ^ε be such that $\theta^\varepsilon(x_1, x_2) = \varepsilon\theta(x_1, x_2)$ for all $(x_1, x_2) \in \overline{\omega}$, where $\theta \in C^2(\overline{\omega})$ is independent of ε.

Denote by $\mathbf{u}(\varepsilon)$ the solution of problem $P(\varepsilon; \Omega)$ which is equivalent to the problem $P(\widehat{\Omega}^\varepsilon)$.

Define the matrix

$$\Lambda^* = \begin{pmatrix} \lambda_1^* & \lambda_3^* \\ \lambda_3^* & \lambda_2^* \end{pmatrix} = \Lambda - \Lambda(\Lambda + 2M)^{-1}\Lambda = 2M(\Lambda + 2M)^{-1}\Lambda.$$

Established following

Theorem 2. *Let, satisfies following conditions*

$$\lambda_4 < 0, \quad \lambda_1 + \frac{2}{3}\mu_1 > 0, \quad \lambda_1^* + \mu_1 > 0, \quad detM > 0, \quad det\left(\Lambda + \frac{2}{3}M\right) > 0.$$

(a) As $\varepsilon \to 0$, the family $(\mathbf{u}(\varepsilon))_{\varepsilon > 0}$ converges strongly in the space

$$\mathbf{V}(\Omega) := \left\{\mathbf{v} \in (H^1(\Omega))^6,\ \mathbf{v} = 0\ \text{ on }\ \Gamma_0\right\},$$

(b) $\mathbf{u} = (u_1', u_2', u_3', u_1'', u_2'', u_3'') := \lim_{\varepsilon \to 0} \mathbf{u}(\varepsilon)$ is such that $u_\alpha = (u_\alpha', u_\alpha'')^T = \zeta_\alpha - x_3\partial_\alpha\zeta_3$ and $u_3 = (u_3', u_3'')^T = (\zeta_3^*, \zeta_3^*)^T$, where $\zeta_j = (\zeta_j', \zeta_j'')^T$, also $\zeta_3' = \zeta_3'' = \zeta_3^*$.

$$\boldsymbol{\zeta} := (\zeta_1', \zeta_2', \zeta_1'', \zeta_2'', \zeta_3^*) \in \mathbf{V}(\omega) := \Big\{\boldsymbol{\eta} = (\eta_1', \eta_2', \eta_1'', \eta_2'', \eta_3^*) \in H^1(\omega)\times$$

$$\times H^1(\omega) \times H^1(\omega) \times H^1(\omega) \times H^2(\omega),\ \ \eta_\alpha' = \eta_\alpha'' = \partial_\nu\eta_3^* = \eta_3^* = 0\ \text{ on }\ \gamma_0\Big\}.$$

(c) The vector field $\boldsymbol{\zeta}$ is a solution the following limit two-dimensional problem $P(\omega)$:

$$\boldsymbol{\zeta} \in \mathbf{V}(\omega)\text{and} - \int\limits_\omega \mathrm{m}_{\alpha\beta}\partial_{\alpha\beta}\eta_3^* \mathrm{d}\omega + \int\limits_\omega \left\{(\mathrm{n}_{\alpha\beta})^{\mathrm{T}}\bar{\mathrm{e}}_{\alpha\beta}(\boldsymbol{\eta}) - 4\lambda_4 \mathrm{h}_{\alpha\beta}(\boldsymbol{\zeta})\mathrm{h}_{\alpha\beta}(\boldsymbol{\eta})\right\} \mathrm{d}\omega$$

$$= \int\limits_\omega (p_\alpha)^T \eta_\alpha d\omega + \int\limits_\omega (p_3' + p_3'')\eta_3^* d\omega - \int\limits_\omega (q_\alpha' + q_\alpha'')\partial_\alpha\eta_3^* d\omega$$

for any $\boldsymbol{\eta} \in \mathbf{V}(\omega)$, *where*

$$
\begin{aligned}
& m_{\alpha\beta} := -\left\{ \frac{2}{3} \widehat{\lambda} \Delta \zeta_3^* \delta_{\alpha\beta} + \frac{4}{3} \widehat{\mu} \partial_{\alpha\beta} \zeta_3^* \right\}, \\
& n_{\alpha\beta} := 2\Lambda^* \bar{e}_{\sigma\sigma}(\zeta) \delta_{\alpha\beta} + 4M \bar{e}_{\alpha\beta}(\zeta), \\
& \widehat{\lambda} := \lambda_1^* + 2\lambda_3^* + \lambda_2^*, \quad \widehat{\mu} := \mu_1 + 2\mu_3 + \mu_2, \\
& \bar{e}_{\alpha\beta} = (\bar{e}_{\alpha\beta}^{'}, \ \bar{e}_{\alpha\beta}^{''})^T := \frac{1}{2} (\partial_\alpha \zeta_\beta + \partial_\beta \zeta_\alpha + \partial_\alpha \theta \partial_\beta \zeta_3 + \partial_\beta \theta \partial_\alpha \zeta_3), \\
& h_{\alpha\beta} = \frac{1}{2} (\partial_\alpha \zeta_\beta^{'} - \partial_\beta \zeta_\alpha^{'} + \partial_\beta \zeta_\alpha^{''} - \partial_\alpha \zeta_\beta^{''}),
\end{aligned}
$$

$$
p_i = (p_i^{'}, \ p_i^{''})^T := \int_{-1}^{1} F_i dx_3 + g_i^+ + g_i^-, \quad g_i^\pm := g_i(\cdot, \ \pm 1),
$$

$$
q_\alpha := \int_{-1}^{1} x_3 F_\alpha dx_3 + g_\alpha^+ - g_\alpha^-.
$$

R E F E R E N C E S

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