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ON THE ASYMPTOTIC BEHAVIOUR AS $t \to \infty$ OF SOLUTIONS OF ONE NONLINEAR INTEGRO- DIFFERENTIAL PARABOLIC EQUATION ARISING IN THE PENETRATION OF A MAGNETIC FIELD INTO A SUBSTANCE

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In work [1] is proposed one new class of nonlinear partial differential equations with the coefficients consisting of the integral from quadrate of the gradient of an unknown function. For describing the process of penetration of the magnetic field into the material conductivity coefficient of which depends on the temperature in [1] the system is proposed which is transformed into the following system of equations

$$\frac{\partial U}{\partial t} + rot\left[a\left(\int_0^t |rotU|^2\, d\tau\right) rotU\right] = f, \qquad divU = 0, \tag{1}$$

where U is a vector-function (U_1, U_2, U_3).

The existence and uniqueness of the solution of initial-boundary value problems for (1) type systems have been studied in numerous works.

This research was thoroughly carried out for a scalar analogue of equation (1)

$$\frac{\partial U}{\partial t} - \nabla\left[a\left(\int_0^t |\nabla U|^2\, d\tau\right) \nabla U\right] = f, \tag{2}$$

where ∇ is the gradient operation (see, for this purpose, e.g. [1]-[8]).

In the work [6] some generalization of equations of type (2) is proposed. In particular, assuming the temperature of the considered body to be constant throughout the material, i.e., depending on time, but independent of the space coordinates, the process of penetration of the magnetic field into the material is modelled by averaged integro-differential equations.

We should note that equation (2) appears in modelling other physical processes. For example, to one-dimensional case of equation (2) we could transform the system of nonlinear partial differential equations describing adiabatic shearing of incompressible fluids with temperature-dependent viscosity [9].

In studying of initial-boundary value problems for equation (2) is of great importance apriori estimation for higher derivatives. In studying the asymptotic behaviour $(t \to \infty)$ of the solutions the main difficulty is getting apriori estimations, independent of the value of T – the period of time, on which the solution of the problem is built. In this direction we should note the work [10], where there is given stabilisation result of solution when $t \to \infty$ for the first boundary value problem in case of homogeneous boundary conditions and homogeneous right side f.

Let's consider the problem

$$\frac{\partial U}{\partial t} - \frac{\partial}{\partial x}\left(a(S)\frac{\partial U}{\partial x}\right) = 0, \tag{3}$$

where

$$S(x,t) = \int_0^t \left|\frac{\partial U}{\partial x}\right|^2 d\tau, \tag{4}$$

or

$$S(t) = \int_0^t \int_0^1 \left|\frac{\partial U}{\partial x}\right|^2 d\tau, \tag{5}$$

$$U(0,t) = U(1,t) = 0, \tag{6}$$

$$U(x,0) = U_0(x). \tag{7}$$

If $a(S) = (1+S)^p$, $0 < p \leq 1$, $U_0 = U_0(x) \in L_2(0,1)$ then it's not difficult to get validity of the following statement

Lemma 1. *For the solutions of the problems (3),(4),(6),(7) and (3),(5)-(7) the following estimate takes place*

$$\|U\| \leq \|U_0\| \exp(-t),$$

where $\|.\|$ is the norm in the space $L_2(0,1)$.

Using Lemma 1 we can show that the following statement is true

Lemma 2. *For the solutions of the problems (3),(4),(6),(7) and (3),(5)-(7) the following relation holds*

$$\exp(t)\left[C_1\|U\|^2 + C_2\left\|\frac{\partial U}{\partial x}\right\|^2 + C_3\left\|\frac{\partial U}{\partial t}\right\|^2\right] + C_4\int_0^t \exp(\tau)\left\|\frac{\partial U}{\partial x}\right\|^2 d\tau \leq C.$$

Here and below C_i, C and c denote the positive constants independent from t.

From the Lemma 2 follows the

Theorem 1. *If $U_0 \in W_2^2(0,1)$, $U_0(0) = U_0(1) = 0$, then for the solutions of the problems (3),(4),(6),(7) and (3),(5)-(7) the following estimate is true*

$$\left\|\frac{\partial U}{\partial x}\right\| + \left\|\frac{\partial U}{\partial t}\right\| \leq C\exp\left(-\frac{t}{2}\right).$$

Note that Lemma 1 guaranties exponential stabilization of the solutions of the problems (3),(4),(6),(7) and (3),(5)-(7) in the norm of the space $L_2(0,1)$, meanwhile Theorem 1 gives also exponential behavior but in the norm of the space $W_2^1(0,1)$.

Result in the Theorem 1 can be improved. Particularly stabilization is achieved for the $\partial U/\partial x$ in $C^1(0,1)$.

For this, at first we receive auxiliary statement

Lemma 3. *For the function S following estimates are true:*

$$c \le 1 + S(x,t) \le C.$$

Now combining (3)-(5), Theorem 1 and Lemma 3 we get validity of the main second statement

Theorem 2. *If $U_0 \in W_2^2(0,1)$, $U_0(0) = U_0(1) = 0$, then for the solutions of the problems (3),(4),(6),(7) and (3),(5)-(7) the following relation holds*

$$\left| \frac{\partial U(x,t)}{\partial x} \right| \le C \exp\left(-\frac{t}{2} \right).$$

Let's consider now equations (3),(4) and (3),(5) with nonhomogeneous condition on one side of the boundary, i.e., instead (6) we will have

$$U(0,t) = 0, \ \ U(1,t) = \psi = \text{const} > 0. \tag{8}$$

Applying multipliers U, U_t, U_{tt} after long and technically complicated transformations we deduce the necessary a priori estimates:

Lemma 4. *The following estimates are true:*

$$c\varphi^{\frac{1}{1+2p}}(t) \le 1 + S \le C\varphi^{\frac{1}{1+2p}}(t),$$

$$c\varphi^{\frac{2p}{1+2p}}(t) \le \frac{d\varphi}{dt} \le C\varphi^{\frac{2p}{1+2p}}(t),$$

where

$$\varphi(t) = 1 + \int_0^t \int_0^1 (1+S)^{2p} \left(\frac{\partial U}{\partial x} \right)^2 dx d\tau.$$

Lemma 5. *The derivative $\partial U/\partial t$ satisfies the inequality*

$$\int_0^1 \left(\frac{\partial U}{\partial t} \right)^2 dx \le C\varphi^{-\frac{2}{1+2p}}(t).$$

Lemma 6. *For $\partial S/\partial x$ the following estimate is true*

$$\int_0^1 \left| \frac{\partial S}{\partial x} \right| dx \le C\varphi^{-\frac{p}{1+2p}}(t).$$

Using Lemmas 4, 5 and 6 we receive following

Theorem 3. *If $U_0 \in W_2^2(0,1)$, $U_0(0) = 0$, $U_0(1) = \psi$, then for the solutions of the problems (3),(4),(7),(8) and (3),(5),(7),(8) the following estimate is true*

$$\left| \frac{\partial U(x,t)}{\partial x} - \psi \right| \le Ct^{-1-p}, \quad t \ge 1.$$

So, we have stabilization of the solutions of the equation (3),(4) and (3),(5) with the homogeneous and nonhomogeneous boundary conditions. In both cases stabilization is given in the norm of the space $C^1(0,1)$, but there is a difference between rates of the convergence.

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