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ABOUT ONE BOUNDARY VALUE PROBLEM FOR NON-SHALLOW CYLINDRICAL SHELLS

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We shall consider the non-shallow cylindrical shells for I.N. Vekua's $N = 0$ approximation [1], [2].

The system of equilibrium equations of the two-dimensional non-shallow cylindrical shells may be written in the following form [3]

$$\begin{aligned}
\mu\Delta u_1 &+ (\lambda+\mu)\partial_1\theta + (\lambda+3\mu)\sum_{s=0}^{\infty}\frac{\varepsilon^{2s+1}}{2s+1}\partial_1 u_3 \\
&+ (\lambda+2\mu)\sum_{s=1}^{\infty}\frac{\varepsilon^{2s}}{2s+1}\partial_{11}u_1 - \mu\sum_{s=0}^{\infty}\frac{\varepsilon^{2s+2}}{2s+1}u_1 + \Phi_1 = 0, \\
\mu\Delta u_2 &+ (\lambda+\mu)\partial_2\theta + \mu\sum_{s=1}^{\infty}\frac{\varepsilon^{2s}}{2s+1}\partial_{11}u_2 + \varepsilon\lambda\partial_2 u_3 + \Phi_2 = 0, \\
\mu\Delta u_3 &+ \mu\sum_{s=1}^{\infty}\frac{\varepsilon^{2s}}{2s+1}\partial_{11}u_3 - (\lambda+3\mu)\sum_{s=0}^{\infty}\frac{\varepsilon^{2s+1}}{2s+1}\partial_1 u_1 \\
&- (\lambda+2\mu)\sum_{s=0}^{\infty}\frac{\varepsilon^{2s+2}}{2s+1}u_3 - \varepsilon\lambda\partial_2 u_2 + \Phi_3 = 0,
\end{aligned} \tag{1}$$

where $\theta = \partial_1 u_1 + \partial_2 u_2$, $\varepsilon = \frac{h}{R}$ is a small parameter, h-the semithickness of the shell, R-the radius of the middle sirface of the cylinder, $\Phi_i (i = 1,2,3)$- the components of external force, $u_i (i = 1,2,3)$- the components of the displacement vector, λ and μ-Lame's constants, x_1 and x_2-isometric coordinates on the cylindrical surface.

Let us try to construct the solutions of the form [4]

$$u_3 = \sum_{k=0}^{\infty}\overset{k}{u}_3\varepsilon^k, \quad u_\alpha = \sum_{k=0}^{\infty}\overset{k}{u}_\alpha\varepsilon^k \quad (\alpha = 1,2). \tag{2}$$

The formal substitution of (2) into (1) shows that the series (2) may satisfy equations (1) if the following equations are fulfilled

$$\begin{aligned}
\mu\Delta\overset{k}{u}_1 &+ (\lambda+\mu)\partial_1\overset{k}{\theta} = \overset{k}{\Phi}_1, \\
\mu\Delta\overset{k}{u}_2 &+ (\lambda+\mu)\partial_2\overset{k}{\theta} = \overset{k}{\Phi}_2, \\
\mu\Delta\overset{k}{u}_3 &= \overset{k}{\Phi}_3 \; (k = 1,2,...),
\end{aligned} \tag{3}$$

where

$$
\begin{aligned}
\overset{k}{\Phi}_1 &= \Phi_1 - (\lambda+3\mu)\sum_{s=0}^{\left[\frac{k-1}{2}\right]} \frac{\partial_1 \overset{k-2s-1}{u}_3}{2s+1} \\
&- (\lambda+2\mu)\sum_{s=1}^{\left[\frac{k}{2}\right]} \frac{\partial_{11} \overset{k-2s}{u}_1}{2s+1} + \mu \sum_{s=0}^{\left[\frac{k-2}{2}\right]} \frac{\overset{k-2s-2}{u}_1}{2s+1}, \\
\overset{k}{\Phi}_2 &= \Phi_2 - \lambda\partial_2 \overset{k-1}{u}_3 - \mu \sum_{s=1}^{\left[\frac{k}{2}\right]} \frac{\partial_{11} \overset{k-2s}{u}_2}{2s+1}, \\
\overset{k}{\Phi}_3 &= \Phi_3 - \mu \sum_{s=1}^{\left[\frac{k}{2}\right]} \frac{\partial_{11} \overset{k-2s}{u}_3}{2s+1} + (\lambda+3\mu)\sum_{s=0}^{\left[\frac{k-1}{2}\right]} \frac{\partial_1 \overset{k-2s-1}{u}_1}{2s+1} \\
&+ (\lambda+2\mu)\sum_{s=0}^{\left[\frac{k-2}{2}\right]} \frac{\overset{k-2s-2}{u}_3}{2s+1} + \lambda\partial_2 \overset{k-1}{u}_2,
\end{aligned}
$$

$$
(k = 0, 1, 2, ...;\ \ \overset{k}{u}_\alpha = \overset{k}{u}_3 = 0,\ \ \text{if k} < 0;\ \ \alpha = 1, 2).
$$

For each fixed k equations (3) coincide with equations of plane theory of elastisity and Poisson.

Let us consider the case, when the middle surface of the body after developing on the plane, is the circle with radius R and let Φ_i are equal to P_i, where $P_i = const$.

Boundary conditions have the following form

$$
\overset{k}{u}_r + i\overset{k}{u}_\theta = i\overset{k}{u}_+ \frac{d\bar{z}}{ds} = 0,\ \ r = R,
$$

$$
\overset{k}{u}_3 = 0,\ \ r = R.
$$

For $k = 0$ approximation this problem is a well known case of the plane theory of elasticity [5] for which we have

$$
\begin{cases}
2\mu \overset{0}{u}_+ = \dfrac{\mu}{\lambda+3\mu}(z\bar{z} - R^2)P_+, \\
2\mu \overset{0}{u}_3 = (z\bar{z} - R^2)\dfrac{P_3}{2}.
\end{cases}
$$

For $k = 1$ approximation we have

$$
\begin{cases}
\mu\Delta \overset{(1)}{u}_+ + 2(\lambda+\mu)\dfrac{\partial \overset{(1)}{\theta}}{\partial \bar{z}} = -\dfrac{2\lambda+3\mu}{4\mu}P_3 z - \dfrac{3}{4}P_3\bar{z}, \\
\mu\Delta \overset{(1)}{u}_3 = 4[Az + \bar{A}\bar{z}],
\end{cases}
$$

where $A = -\dfrac{(2\lambda+3\mu)\bar{P}_+ + 3\mu P_+}{16(\lambda+3\mu)}$.

The general solutions of these equations have the following forms:

$$\begin{cases} 2\mu \overset{(1)}{u}_{+} = æ\overset{1}{\varphi}(z) - z\overline{\overset{1}{\varphi}{}'(z)} - \overline{\overset{1}{\psi}(z)} - \dfrac{2\lambda+3\mu}{16(\lambda+2\mu)}P_3 z^2\bar{z} \\ \qquad -\dfrac{3(\lambda+3\mu)}{32(\lambda+2\mu)}P_3 z\bar{z}^2 + \dfrac{\lambda+\mu}{32(\lambda+2\mu)}P_3 z^3, \\ 2\mu \overset{(1)}{u}_{3} = \overset{1}{f}(z) + \overline{\overset{1}{f}(z)} + Az^2\bar{z} + \bar{A}z\bar{z}^2. \end{cases}$$

If the functions $\overset{1}{\varphi}(z)$, $\overset{1}{\psi}(z)$, $\overset{1}{f}(z)$ are introduced by series

$$\overset{1}{\varphi}(z) = \sum_{m=1}^{\infty} \overset{1}{a}_{m} z^m, \quad \overset{1}{\psi}(z) = \sum_{m=0}^{\infty} \overset{1}{b}_{m} z^m, \quad \overset{1}{f}(z) = \sum_{m=0}^{\infty} \overset{1}{c}_{m} z^m,$$

and are subtituted in the boundary conditions we have

$$\begin{aligned} &\overset{1}{a}_{1} = \frac{(2\lambda+3\mu)(\lambda+\mu)}{32\mu(\lambda+2\mu)}R^2P_3, \ \ \overset{1}{a}_{3} = -\frac{(\lambda+\mu)^2P_3}{32(\lambda+2\mu)(\lambda+3\mu)}, \\ &\overset{1}{b}_{1} = -\frac{3\mu}{8(\lambda+3\mu)}R^2P_3, \ \ \overset{1}{c}_{1} = -AR^2. \end{aligned}$$

For $k = 2$ approximation we have

$$\begin{cases} \mu\Delta \overset{2}{u}_{+} + 2(\lambda+\mu)\partial_{\bar{z}} \overset{2}{\theta} = D_1 + D_2 z^2 + D_3 z\bar{z} + D_4 \bar{z}^2, \\ \mu\Delta \overset{2}{u}_{3} = -2B_1 - 6B_2(z^2+\bar{z}^2) - 8B_3 z^2\bar{z}^2, \end{cases}$$

where

$$\begin{aligned} D_1 &= \left(\frac{2\lambda+3\mu}{2\mu}\bar{A} + \frac{3}{2}A\right)R^2 - \frac{P_+}{6} - \frac{\lambda+\mu}{6(\lambda+3\mu)}\bar{P}_+, \\ D_2 &= -\frac{2\lambda+3\mu}{2\mu}A, \ D_3 = -\frac{2\lambda+3\mu}{2\mu}\bar{A} - 3A, \quad D_4 = -\frac{3}{2}\bar{A}, \\ B_1 &= -\left(\frac{(2\lambda+3\mu)^2}{64(\lambda+2\mu)} + \frac{9\mu}{32(\lambda+3\mu)} - \frac{\lambda+2\mu}{8\mu} - \frac{1}{12}\right)R^2P_3, \\ B_2 &= \frac{3(2\lambda+3\mu)(3\lambda+7\mu)}{256(\lambda+2\mu)(\lambda+3\mu)}P_3, \\ B_3 &= \left(\frac{(2\lambda+3\mu)^2}{128\mu(\lambda+2\mu)} + \frac{9\mu}{64(\lambda+3\mu)} - \frac{\lambda+2\mu}{32\mu}\right)P_3. \end{aligned}$$

The general solutions of these equations have the following forms

$$\begin{cases} 2\mu \overset{(2)}{u}_{+} = æ\overset{2}{\varphi}(z) - z\overline{\overset{2}{\varphi}{}'(z)} - \overline{\overset{2}{\psi}(z)} - A_1 z\bar{z} - A_2 z^2 - A_3\bar{z}z^3 - A_4 z^2\bar{z}^2 - A_5 z\bar{z}^3 - A_6 z^4, \\ 2\mu \overset{(2)}{u}_{3} = \overset{2}{f}(z) + \overline{\overset{2}{f}(z)} - B_1 z\bar{z} - B_2(\bar{z}z^3 + z\bar{z}^3) - B_3 z^2\bar{z}^2, \end{cases}$$

where

$$A_1 = -\frac{\lambda+3\mu}{4(\lambda+2\mu)}D_1, \quad A_2 = \frac{\lambda+\mu}{8(\lambda+2\mu)}\bar{D}_1, \; A_3 = -\frac{1}{6}\left(\frac{\lambda+3\mu}{2(\lambda+2\mu)}D_2 - \frac{\lambda+\mu}{4(\lambda+2\mu)}\bar{D}_3\right),$$
$$A_4 = -\frac{1}{4}\left(\frac{\lambda+3\mu}{4(\lambda+2\mu)}D_3 - \frac{\lambda+\mu}{2(\lambda+2\mu)}\bar{D}_2\right), \; A_5 = -\frac{\lambda+3\mu}{12(\lambda+2\mu)}D_4, \quad A_6 = \frac{\lambda+\mu}{48(\lambda+2\mu)}\bar{D}_4.$$

If the functions $\overset{2}{\varphi}(z), \;\; \overset{2}{\psi}(z), \;\; \overset{2}{f}(z)$ are introduced by series

$$\overset{2}{\varphi}(z) = \sum_{m=1}^{\infty} \overset{2}{a}_m z^m, \quad \overset{2}{\psi}(z) = \sum_{m=0}^{\infty} \overset{2}{b}_m z^m, \quad \overset{2}{f}(z) = \sum_{m=0}^{\infty} \overset{2}{c}_m z^m,$$

and are subtituted in the boundary conditions we have

$$\begin{aligned} \overset{2}{a}_2 &= \frac{A_2 + A_3R^2}{æ}, \;\; \overset{2}{a}_4 = \frac{A_6}{æ}, \;\; \overset{2}{b}_0 = -\bar{A}_1R^2 - \bar{A}_4R^4 - \frac{2(A_2 + A_3R^2)}{æ}R^2, \\ \overset{2}{b}_2 &= -\bar{A}_5R^2 - \frac{4A_6}{æ}R^2, \;\; \overset{2}{c}_0 + \overline{\overset{2}{c}}_0 = B_1R^2 + B_3R^4, \;\; \overset{2}{c}_3 = B_2R^2. \end{aligned}$$

For the components of the displacements we obtain

$$\begin{aligned} u_r &= \frac{1}{2\mu}\left\{\frac{\mu}{\lambda+3\mu}(r^2-R^2)(P_1cos\theta + P_2sin\theta)\right. \\ &- \varepsilon\left[\frac{2\lambda+3\mu}{16(\lambda+2\mu)}(r^2-R^2)rP_3 + \frac{3\mu}{8(\lambda+3\mu)}(r^2-R^2)rP_3cos2\theta\right] \\ &+ \varepsilon^2\left[Re\left((R^2-r^2)r^2A_3 - \left(2\overline{\overset{2}{a}}_2 + A_1\right)r^2 - A_4r^4 - \overline{\overset{2}{b}}_0\right)cos\theta\right. \\ &- Im\left((R^2-r^2)r^2A_3 + \left(2\overline{\overset{2}{a}}_2 + A_1\right)r^2 + A_4r^4 + \overline{\overset{2}{b}}_0\right)sin\theta \\ &- \left.\left. Re\left(\overline{\overset{2}{b}}_2r^2 + \left(4\overline{\overset{2}{a}}_4 + A_5\right)r^4\right)cos3\theta - Im\left(\overline{\overset{2}{b}}_2r^2 + \left(4\overline{\overset{2}{a}}_4 + A_5\right)r^4\right)sin3\theta\right]\right\}, \\ u_\theta &= \frac{1}{2\mu}\left\{\frac{\mu}{\lambda+3\mu}(r^2-R^2)(P_2cos\theta - P_1sin\theta) - \varepsilon\left[\frac{3\mu}{8(\lambda+3\mu)}(r^2-R^2)rP_3sin2\theta\right]\right. \\ &+ \varepsilon^2\left[Im\left((R^2-r^2)r^2A_3 - \left(2\overline{\overset{2}{a}}_2 + A_1\right)r^2 - A_4r^4 - \overline{\overset{2}{b}}_0\right)cos\theta\right. \\ &+ Re\left((R^2-r^2)r^2A_3 + \left(2\overline{\overset{2}{a}}_2 + A_1\right)r^2 + A_4r^4 + \overline{\overset{2}{b}}_0\right)sin\theta \\ &- \left.\left. Im\left(\overline{\overset{2}{b}}_2r^2 + \left(4\overline{\overset{2}{a}}_4 + A_5\right)r^4\right)cos3\theta + Re\left(\overline{\overset{2}{b}}_2r^2 + \left(4\overline{\overset{2}{a}}_4 + A_5\right)r^4\right)sin3\theta\right]\right\}, \\ u_3 &= \frac{1}{2\mu}\left\{\frac{r^2-R^2}{2}P_3 + \varepsilon\left[2(r^3 - rR^2)(ReAcos\theta + ImAsin\theta)\right]\right. \\ &+ \left.\varepsilon^2\left[B_1(R^2-r^2) + B_3(R^4-r^4) + 2B_2r^2(R^2-r^2)cos2\theta\right]\right\}. \end{aligned}$$

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