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INITIAL BOUNDARY VALUE PROBLEM FOR ONE GENERALIZATION OF SCHRÖDINGER EQUATION

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In the present paper we study initial boundary value problem for Schrödinger type equation, where the unknown function depend on several time variables. The nonclassical equations with several time variables can be considered as mathematical models of physical, ecological, technological and other processes. Particularly, multi-time parabolic equations, which are also called pluriparabolic or ultraparabolic equations, describe the Brown's motion of particle [1], the processes of refining of impurities of Silicon laminae [2], diffusion processes [3] and impurity spreading in rivers [4,5]. Schrödinger type equations with several time variables arise while mathematical modelling of the nonstationary stimulated Raman scaterring [6]. Many interesting works are devoted to the investigation of classical and nonclassical initial boundary value problems for multi-time evolution equations ([7-15]).

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary $\Gamma = \partial\Omega$. We denote by $H^s(\Omega), s > 0,\ s \in \mathbb{R}$, the Sobolev space of complex-valued function of order s based on $L^2(\Omega)$. $H_0^s(\Omega)$ is the closure of the set $\mathcal{D}(\Omega)$ of infinitely differentiable functions with compact support in Ω in the space $H^s(\Omega)$. For any complex Banach space V we denote by $L^2(\prod_{j=1}^{3}(0,T_j);V)$ the space of vector functions defined in three dimensional parallelepiped $(0,T_1)\times(0,T_2)\times(0,T_3)$ with values in V such that $\|g\|_V \in L^2(\prod_{j=1}^{3}(0,T_j))$. Each vector function $g \in L^2(\prod_{j=1}^{3}(0,T_j);V)$ we identify with distribution on $\prod_{j=1}^{3}(0,T_j)$ ranging in V [16], and we denote its generalized derivatives by $\partial g/\partial t_k \in \mathcal{D}'(\prod_{j=1}^{3}(0,T_j);V),\ k=1,2,3$. Let A be an elliptic operator from $H_0^1(\Omega)$ to its antidual $H^{-1}(\Omega)$ of the following form

$$Av \equiv -\sum_{j,p=1}^{n}\frac{\partial}{\partial x_j}\left(a_{jp}\frac{\partial v}{\partial x_p}\right)+a_0 v, \qquad \forall v \in H_0^1(\Omega),$$

where $a_{jp}, a_0 \in L^\infty(\Omega),\ a_{jp}(x) = \overline{a_{pj}(x)},\ a_0(x) \geq 0$, for almost all $x \in \Omega$, $\partial/\partial x_j$ denotes the generalized derivative with respect to $x_j,\ j,p=\overline{1,n}$, and $\sum_{j,p=1}^{n} a_{jp}(x)\xi_p\overline{\xi_j} \geq$

$\alpha \sum_{j=1}^{n} |\xi_j|^2$, $\alpha = const > 0$, for all $(\xi_1, ..., \xi_n) \in \mathbb{C}^n$, $\bar{y}$ is the complex conjugate and $|y|$ is the modulus of a complex number $y \in \mathbb{C}$.

Let us consider initial boundary value problem with homogeneous Dirichlet boundary conditions for the following generalization of Schrödinger equation

$$\frac{\partial u}{\partial t_1} + \frac{\partial u}{\partial t_2} + \frac{\partial u}{\partial t_3} + iAu = f, \qquad (t_1, t_2, t_3) \in (0, T_1) \times (0, T_2) \times (0, T_3),$$

which admits the following variational formulation: find the unknown function $u \in L^2(\prod_{j=1}^{3}(0, T_j); H_0^1(\Omega))$, $\partial u / \partial t_k \in L^2(\prod_{j=1}^{3}(0, T_j); H^{-1}(\Omega))$, $k = 1, 2, 3$, which satisfies the equation

$$\sum_{j=1}^{3} \frac{\partial}{\partial t_j}(u(.), v)_{L^2(\Omega)} + i \int_{\Omega} \left(\sum_{j,p=1}^{n} a_{jp} \frac{\partial u(.)}{\partial x_p} \frac{\partial \overline{v}}{\partial x_j} + a_0 u(.) \overline{v} \right) dx = \langle f(.), v \rangle, \qquad (1)$$

for all $v \in H_0^1(\Omega)$ in the sense of distributions on $\prod_{j=1}^{3}(0, T_j)$ and the initial conditions

$$\begin{array}{ll} u(0, t_2, t_3) = \varphi_1(t_2, t_3), & (t_2, t_3) \in (0, T_2) \times (0, T_3), \\ u(t_1, 0, t_3) = \varphi_2(t_1, t_3), & (t_1, t_3) \in (0, T_1) \times (0, T_3), \\ u(t_1, t_2, 0) = \varphi_3(t_1, t_2), & (t_1, t_2) \in (0, T_1) \times (0, T_2), \end{array} \qquad (2)$$

where $f, \varphi_1, \varphi_2, \varphi_3$ are given functions from suitable spaces, $\langle ., . \rangle$ denotes the antiduality relation between the spaces $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. Note that it can be defined traces of function u on the sides of the parallelepiped $(0, T_1) \times (0, T_2) \times (0, T_3)$ and the k-th ($k = \overline{1, 2, 3}$) condition of (2) can be considered as equality in the space $L^2(\prod_{j \neq k}(0, T_j); L^2(\Omega))$.

For the stated problem (1), (2) the following theorem is valid.

Theorem. *If* $\varphi_k \in L^2(\prod_{j \neq k}(0, T_j); H_0^1(\Omega))$, $\partial \varphi_k / \partial t_q \in L^2(\prod_{j \neq k}(0, T_j); H^{-1}(\Omega))$, $1 \leq q \leq 3$, $q \neq k$, $\varphi_1(0, t_3) = \varphi_2(0, t_3)$, $\varphi_2(t_1, 0) = \varphi_3(t_1, 0)$, $\varphi_1(t_2, 0) = \varphi_3(0, t_2)$, $(t_1, t_2, t_3) \in \prod_{j=1}^{3}(0, T_j)$ *and* $f, \partial f / \partial t_k \in L^2(\prod_{j=1}^{3}(0, T_j); H^{-1}(\Omega))$, $k = 1, 2, 3$, *then initial boundary value problem for the generalized Schrödinger equation has a unique solution and the following estimate is valid*

$$\|u\|^2_{L^2(\prod_{j=1}^{3}(0,T_j); H_0^1(\Omega))} + \sum_{k=1}^{3} \left\| \frac{\partial u}{\partial t_k} \right\|^2_{L^2(\prod_{j=1}^{3}(0,T_j); H^{-1}(\Omega))} \leq c \left(\sum_{k=1}^{3} \left(\|\varphi_k\|^2_{L^2(\prod_{j \neq k}(0,T_j); H_0^1(\Omega))} + \right.\right.$$

$$\left.\left. + \sum_{q=1,\, q \neq k}^{3} \left\| \frac{\partial \varphi_k}{\partial t_q} \right\|^2_{L^2(\prod_{j \neq k}(0,T_j); H_0^1(\Omega))} + \left\| \frac{\partial f}{\partial t_k} \right\|^2_{L^2(\prod_{j=1}^{3}(0,T_j); H^{-1}(\Omega))} \right) + \|f\|^2_{L^2(\prod_{j=1}^{3}(0,T_j); H^{-1}(\Omega))} \right).$$

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