Reports of Enlarged Session of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 20, №2, 2005

# ABOUT ONE NON-CLASSICAL PROBLEM IN THEORY OF ELASTICITY

Giorgashvili L. Meladze R.

Georgian Technical University
I. Vekua Institute of Applied Mathematics

## *Abstract*

The statics boundary value problem of theory of elasticity is considered, when it is given the ultimate the normal constituent of displacement vector and the tangent constituent of rotation.

It is proved, that the problem has a singular solution. The solution is presented in the form of absolutely and uniformly convergent series.

*Keywords and phrases*: Isotropy, elasticity, Green's formula, displacement.

*AMS subject classification* (2000): 35J55; 75H20.

The homogeneous system of differential equations of statics for isotropic elastic solid reeds as follows [3]:

$$A(\partial x)u(x) = \mu\Delta u(x) + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,u(x), \tag{1}$$

where $\Delta$ is the Laplace operator, $u=(u_1,u_2,u_3)^\top$ is the displacement vector, $\lambda$ and $\mu$-Lame constants satisfy the in equality $\mu>0,\ \ 3\lambda+2\mu>0$, $\top$ is the symbol of transposition.

Let $\Omega^+$ be the sphere bounded by spherical surface $\partial\Omega$ with the center in the origin and radius $R$, $\Omega^- = R^3\ \overline{\Omega^+}$.

**Problem** $(N)^\pm$. Find a regular $(u\in C^2(\Omega^\pm)\cap C^1(\overline{\Omega^\pm}))$ solution $u(x)$ of the system (1) in the domain $\Omega^+(\Omega^-)$ satisfying the following condition on the boundary $\partial\Omega$

$$[n(z)\cdot u(z)]^\pm = f_4(z), \quad [n(z)\times \mathrm{rot}\,u(z)]^\pm = f(z), \quad z\in\partial\Omega \tag{2}$$

where $f(z) = (f_1(z), f_2(z), f_3(z))^\top, \quad f_j(z), \quad j=1,2,3,4$ are the given functions on the boundary $\partial\Omega$, $n(z)$ is the out ward unit normal vector to $\Omega^+$ at the point $z\in\partial\Omega$.

In case of problem $(N)^-$ in the vicinity of infinity the vector $u(x)$ satisfies the following conditions:

$$u_j(x) = O(|x|^{-1}), \quad \frac{\partial u_j(x)}{\partial x_k} = o(|x|^{-1}), \quad k,j=1,2,3. \tag{3}$$

**Theorem .** *The problem $(N)^\pm$ permits not more that one regular solution.*

**Proof.** The theorem will be proved, if we show, that the corresponding homogeneous problem $(N)_0^\pm$ $(f_j=0,\ j=1,2,3,4)$ has only trivial solution. as

$$u\cdot\Delta v = \mathrm{div}(u\,\mathrm{div}\,v) - \mathrm{div}\,u\,\mathrm{div}\,v + \mathrm{div}[u\times\mathrm{rot}\,v] - \mathrm{rot}\,u\cdot\mathrm{rot}\,v,$$
$$u\cdot\mathrm{grad}\,\mathrm{div}\,v = \mathrm{div}(u\,\mathrm{div}\,v) - \mathrm{div}\,u\,\mathrm{div}\,v,$$

where $u=(u_1,\ u_2,\ u_3)^\top, \quad v=(v_1,\ v_2,\ v_3)^\top$ are thereee-dimensional vectors, therefore

$$u \cdot A(\partial x)u = \operatorname{div}[(\lambda+2\mu)u \operatorname{div} u + \mu(u \times \operatorname{rot} u)]$$
$$-[(\lambda+2\mu)(\operatorname{div} u)^2 + \mu(\operatorname{rot} u)^2].$$

applying Gauss-Ostrogradski formula to the last equality, we'll have

$$\int\limits_{\Omega^+} u \cdot A(\partial x)u \, dx = \int\limits_{\partial\Omega} [u(z)]^+ \cdot [P(\partial z, n)u(z)]^+ \, ds - \int\limits_{\Omega^+} \widetilde{E}(u,u) \, dx, \tag{4}$$

where

$$P(\partial x, n)u(x) = (\lambda+2\mu)n \operatorname{div} u(x) - \mu[n \times \operatorname{rot} u(x)],$$
$$\widetilde{E}(u,u) = (\lambda+2\mu)(\operatorname{div} u)^2 + \mu(\operatorname{rot} u)^2.$$

We'll have the similar formula for the $\Omega^-$ domain

$$\int\limits_{\Omega^-} u \cdot A(\partial x)u \, dx = -\int\limits_{\partial\Omega} [u(z)]^- \cdot [P(\partial z, n)u(z)]^- \, ds - \int\limits_{\Omega^-} \widetilde{E}(u,u) \, dx, \tag{5}$$

From the homogeneous boundary conditions (2)($f_j = 0, \;\; j = 1,2,3,4$) follows

$$[u(z)]^\pm \cdot [P(\partial z, n)u(z)]^\pm = 0. \tag{6}$$

Keeping in mind equalities (1) and (6) in formulas (4) and (5),we'll have.
Then we'll have

$$\int\limits_{\Omega^\pm} \widetilde{E}(u,u) \, dx = 0$$

and

$$\operatorname{div} u(x) = 0, \quad \operatorname{rot} u(x) = 0, \quad x \in \Omega^\pm.$$

$$u(x) = \operatorname{grad} \Phi(x), \quad \Delta\Phi(x) = 0, \quad x \in \Omega^\pm. \tag{7}$$

as $[n(z) \cdot u(z)]^\pm = 0$, therefore the function $\Phi(x)$ satisfies the Neiman condition on boundary $\partial\Omega$ and $\Phi(x) = C = const \quad x \in \Omega^\pm$.

If we use this value of function $\Phi(x)$ in (7),we'll have $u(x) = 0$, $x \in \Omega^\pm$. □

The solution of the problem we are looking in the form [1]

$$u(x) = \operatorname{grad} \Phi_1(x) - a \operatorname{grad} r^2 \Big(r\frac{\partial}{\partial r} + 1\Big)\Phi_2(x) \tag{8}$$
$$+ \operatorname{rot}\operatorname{rot}(xr^2\Phi_2(x)) + \operatorname{rot}(x\Phi_3(x)),$$

where

$$a = \mu(\lambda+2\mu)^{-1} \quad \Delta\Phi_j(x) = 0, \;\; j = 1,2,3, \;\; r = |x|, \;\; r\frac{\partial}{\partial r} = x \cdot \operatorname{grad}.$$

We write down the functions $\Phi_j(x), \quad j = 1,2,3$ in the form

$$\Phi_j(x) = \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \Big(\frac{r}{R}\Big)^k Y_k^{(m)}(\theta,\varphi) A_{mk}^{(j)}, \quad j = 1,2,3, \tag{9}$$

where $A_{mk}^{(j)}$-desired constants, $(r,\theta,\varphi)$-spherical coordinated of the point $x$,

$$Y_k^{(m)}(\theta,\varphi)=\sqrt{\frac{2k+1}{4\pi}\cdot\frac{(k-m)!}{(k+m)!}}P_k^{(m)}(\cos\theta)e^{im\varphi},$$

$P_k^{(m)}(\cos\theta)$-Lejandr's added function.

If the value of functions $\Phi_j(x)$ put from (9) to (8), we'll have:

$$\begin{aligned}u^{(j)}(x)=\sum_{k=o}^{\infty}\sum_{m=-k}^{k}\{&u_{mk}(r)X_{mk}(\theta,\varphi)+\sqrt{k(k+1)}[v_{mk}(r)Y_{mk}(\theta,\varphi)\\&+w_{mk}(r)Z_{mk}(\theta,\varphi)]\},\end{aligned}\tag{10}$$

where

$$\begin{aligned}u_{mk}(r)&=\frac{k}{R}\Big(\frac{r}{R}\Big)^{k-1}A_{mk}^{(1)}+R(k+1)(kb-2a)\Big(\frac{r}{R}\Big)^{k+1}A_{mk}^{(2)},\ \ k\geq 0,\\ v_{mk}(r)&=\frac{1}{R}\Big(\frac{r}{R}\Big)^{k-1}A_{mk}^{(1)}+R\Big[b(k+1)+2\Big]\Big(\frac{r}{R}\Big)^{k+1}A_{mk}^{(2)},\ \ k\geq 1,\\ w_{mk}(r)&=\Big(\frac{r}{R}\Big)^{k}A_{mk}^{(3)},\ \ k\geq 1,\ \ b=1-a,\end{aligned}$$

$\{X_{mk}(\theta,\varphi),\ Y_{mk}(\theta,\varphi),\ Z_{mk}(\theta,\varphi),\}_{|m|\leq k,\ \overline{k=0,\infty}}$ is the full system of ortonormalized vectors in $L_2(\Sigma_1)$ class.[4,5]

From (10) we'll have

$$n(x)\times\operatorname{rot}u(x)=\sum_{k=1}^{\infty}\sum_{m=-k}^{k}\sqrt{k(k+1)}[v_{mk}^{(1)}(r)Y_{mk}(\theta,\varphi)+w_{mk}^{(1)}(r)Z_{mk}(\theta,\varphi)]\tag{11}$$

where

$$\begin{aligned}v_{mk}^{(1)}(r)&=-2(2k+3)\Big(\frac{r}{R}\Big)^{k}A_{mk}^{(2)},\\ w_{mk}^{(1)}(r)&=-\frac{k+1}{R}\Big(\frac{r}{R}\Big)^{k-1}A_{mk}^{(3)},\ \ k\geq 1.\end{aligned}$$

Let's spread out the function $f_4(z)$ and the vector $f(z)$ by Fourie-Laplass series. as $n(z)\cdot f(z)=0$, we'll have

$$f(z)=\sum_{k=1}^{\infty}\sum_{m=-k}^{k}\sqrt{k(k+1)}[\beta_{mk}Y_{mk}(\theta,\varphi)+\gamma_{mk}Z_{mk}(\theta,\varphi)],\tag{12}$$

$$f_4(z)=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\alpha_{mk}Y_k^{(m)}(\theta,\varphi).$$

proceeding to limit in both sides of (10) and (12) equalities, we'll have the following system for desired constants:

$$\begin{aligned}&\frac{k}{R}A_{mk}^{(1)}+R(k+1)(bk-2a)A_{mk}^{(2)}=\alpha_{mk},\ \ k\geq 0,\\ &2(2k+3)A_{mk}^{(2)}=-\beta_{mk},\ \ k\geq 1,\\ &\frac{k+1}{R}A_{mk}^{(3)}=-\gamma_{mk},\ \ k\geq 1.\end{aligned}$$

Putting the solutions of this system in (10), we'll have

$$\begin{aligned} u(x) = \sum_{k=o}^{\infty} \sum_{m=-k}^{k} \left(\frac{r}{R}\right)^{k+1} \Big\{ \alpha_{mk} X_{mk}(\theta,\varphi) + \sqrt{k(k+1)} \Big[ \Big( \frac{1}{k}\alpha_{mk} \\ - \frac{R(a(k+1)+k)}{k(2k+3)} \beta_{mk} \Big) Y_{mk}(\theta,\varphi) - \frac{R^2}{r(k+1)} \gamma_{mk} Z_{mk}(\theta,\varphi) \Big] \Big\} \\ + \frac{R^2-r^2}{R^3} \sum_{k=1}^{\infty} \sum_{m=-k}^{k} \left(\frac{r}{R}\right)^{k-1} \Big[ kX_{mk}(\theta,\varphi) + \sqrt{k(k+1)} Y_{mk}(\theta,\varphi) \Big] A_{mk}^{(1)} \end{aligned} \tag{13}$$

The series (13) and its derivative series will be absolutely and uniformly convergent in the domain $\overline{\Omega^+}$, if

$$\alpha_{mk} = O(k^{-4}), \ \ \beta_{mk} = O(k^{-4}), \ \ \gamma_{mk} = O(k^{-4}). \tag{14}$$

we'll have this estimates, if [2]

$$f_4(z) \in C^4(\partial\Omega), \quad f(z) \in C^3(\partial\Omega).$$

## REFERENCES


1. Giorgashvili G. Solution of the basic boundary value problems of statics theory of elasticity for the sphere the works of. IAM, 1981, 32-36.

2. Giorgashvili G. Solution of the basic boundary value problems of stationary thermoelastic oscillations for domains bounded by spherical surfaces, Georgian Math. J. 4 (1997), N 5, p. 421–438.

3. Kupradze V.D., Gegelia T.G., Basheleishvili M.O., Burchuladze T.V. Three-dimensional problems of the mathematical theory of elasticity and thermoelasticity, Nauka, Moscow, 1976.

4. Mors F.M., Feshbakh G. Methods of Theoretical Physics, Foreign Literature Publishing House. Moscow. 2. 1980.

5. Ulitko A.F. The method of eigenvector functions in three-dimensional problems of the theory of elasticity, Naukova Dumka, Kiev, 1979.


Received 29. IV. 2005; accepted 10. VI. 2005.