Reports of Enlarged Session of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 20, №2, 2005

SOLUTION OF THE FIRST BOUNDARY VALUE PROBLEM OF STATICS OF THE THEORY OF THERMOELASTIC MIXTURE FOR A DISC

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Equations of statics of the theory of a thermoelastic mixture have the following form [1]:

$$\begin{aligned} &a_1\Delta u^{(1)} + b_1 \mathrm{graddiv} u^{(1)} + c\Delta u^{(2)} + d\,\mathrm{graddiv} u^{(2)} = \gamma_1 \mathrm{grad} u_3,\\ &c\Delta u^{(1)} + d\,\mathrm{graddiv} u^{(1)} + a_2\Delta u^{(2)} + b_2 \mathrm{graddiv} \mathrm{u}^{(2)} = \gamma_2 \mathrm{grad} \mathrm{u}_3,\\ &\Delta u_3 = 0, \end{aligned} \tag{1}$$

where $u^{(i)} = (u_1^{(i)}, u_2^{(i)})$ are the vectors of particular displacements; u_3 is the scalar function denoting temperature variation; a_1, a_2, b_1, b_2, c, d are elastic constants of the mixture; γ_1, γ_2 are temperature constants, $i = 1, 2$.

Consider the disc $D(0, R)$ of radius R with boundary C, occupied by the elastic mixture.

Problem. Find in the disc D a regular solution $U(x) = (u^{(i)}, u_3)$ satisfying the boundary conditions

$$u^{(i)}(z) = f^{(i)}(z), \tag{2}$$

$$\frac{du_3(z)}{dn} = f_3(z), \tag{3}$$

where $x = (x_1, x_2) \in D$, $z \in C$, $f^{(i)} = (f_1^{(i)}, f_2^{(i)})$, $\frac{du_3}{dn}$ is the heat flow, $n = (n_1, n_2)$ is the outer normal to the circumference C; $f_1^{(i)}, f_2^{(i)}, f_3$ are the continuous functions given on C, and

$$\int_C f_3(y)\, d_yC = 0. \tag{4}$$

Suppose that the function f_3 expands into a Fourier series. The unknown function $u_3(x)$, as a solution of the Neumann problem for the Laplace equation with the condition (3), with regard for (4), can be represented in the form [2]

$$u_3(x) = \sum_{m=1}^{\infty} \frac{R}{m} f_{3m}(x) + K_0, \tag{5}$$

or in the form of Dini's formula

$$u_3(x) = \frac{1}{\pi} \int_C f_3(y) \ln \frac{1}{\rho}\, d_yC + K_0, \tag{6}$$

where $x = (r;\psi)$, $z = (R,\psi) \in C$, $y = (R,\theta) \in C$, $\rho = |x-y|$, $r^2 = x_1^2 + x_2^2$, $x_1 = r\cos\psi$, $x_2 = r\sin\psi$, $y_1 = R\cos\theta$, $y_2 = R\sin\theta$,

$$f_{3m}(x) = \frac{1}{\pi}\left(\frac{r}{R}\right)^m \int_0^{2\pi} \cos m(\psi-\theta) f_3(\theta)\,d\theta \tag{7}$$

is a homogeneous harmonic function of the m-th order, K_0 is an arbitrary constant.

A solution $u^{(i)} = (u_1^{(i)}, u_2^{(i)})$ will be sought in the form of the sum of two vectors

$$u^{(i)}(x) = w_1^{(i)} + w_2^{(i)}, \quad i = 1,2, \tag{8}$$

where $w_1^{(i)}$ is the solution of the problem

$$\begin{aligned} &a_1\Delta w_1^{(1)} + b_1 \text{graddiv} w_1^{(1)} + c\Delta w_1^{(2)} + d\text{graddiv} w_1^{(2)} = \gamma_1 \text{grad} u_3,\\ &c\Delta w_1^{(1)} + d\text{graddiv} w_1^{(1)} + a_2\Delta w_1^{(2)} + b_2 \text{graddiv} w_1^{(2)} = \gamma_2 \text{grad} u_3, \end{aligned} \tag{9}$$

$$\left\{w_1^{(i)}(x)\right\}_{r=R} = 0, \tag{10}$$

and $w_2^{(i)}$ is the solution of the problem

$$\begin{aligned} &a_1\Delta w_2^{(1)} + b_1 \text{graddiv} w_2^{(1)} + c\Delta w_2^{(2)} + d\text{graddiv} w_2^{(2)} = 0,\\ &c\Delta w_2^{(1)} + d\text{graddiv} w_2^{(1)} + a_2\Delta w_2^{(2)} + b_2 \text{graddiv} w_2^{(2)} = 0, \end{aligned} \tag{11}$$

$$\left\{w_2^{(i)}(x)\right\}_{r=R} = f^{(i)}(z), \quad z \in C. \tag{12}$$

A solution $w_1^{(i)}(x)$ of the problem (9)–(10) we seek in the form

$$w_1^{(i)}(x) = (R^2 - r^2)\text{grad} \sum_{\text{m}=1}^{\infty} \alpha_{\text{m}}^{(\text{i})} \text{f}_{3\text{m}}(\text{x}), \quad \text{i} = 1,2, \tag{13}$$

where $\alpha_m^{(i)}$ are unknown constants, and f_{3m} is defined by formula (7).

In equation (9) we substitute the expressions (5) and (13). For every m we obtain a system of equations with respect to the unknowns $\alpha_m^{(i)}$. Solving this system and substituting the obtained solutions in (13), we find that

$$w_1^{(i)}(x) = \frac{(R^2-r^2)Re_i}{\pi b} \text{grad} \int_0^{2\pi} \sum_{\text{m}=1}^{\infty} \frac{1}{\text{m}^2}\left(\frac{\text{r}}{\text{R}}\right)^{\text{m}} \cos \text{m}(\psi-\theta) \text{f}_3(\theta)\,\text{d}\theta,$$

where $e_1 = -\dfrac{\gamma_1 b + 2e_2(2c+d)}{2(2a_1+b)}$, $e_2 = \gamma_2(2a_1+1) - \gamma_1(2c+d)$, $b = 2[(2c+d)^2 - (2a_1+b_1)(2a_2+b_2)]$.

Using the equalities $\sum\limits_{m=1}^{\infty} \frac{\tau^m}{m} = -\ln(1-\tau)$, $|\tau| < 1$ [5], we can obtain [6]

$$\sum_{m=1}^{\infty} \frac{1}{m^2}\left(\frac{r}{R}\right)^m \cos m(\psi-\theta) = \int_0^1 \frac{1}{t} \ln\frac{R}{R_1}\,dt \equiv M_1(x,y), \tag{14}$$

where $\tau = \frac{\xi}{\zeta}\, t$, $t \in [0,1]$, $\xi = re^{i\psi}$, $\zeta = Re^{i\theta}$, $R_1^2 = R^2 + t^2r^2 - 2Rrt\cos(\psi - \theta)$.

Thus

$$w_1^{(i)}(x) = \frac{(R^2 - r^2)e_i}{\pi b} \operatorname{grad} \int_C \mathrm{M_1(x,y)f_3(y)\,d_yC}, \tag{15}$$

where $d_yC = R\,d\theta$, $i = 1, 2$.

To solve the problem (11)–(12) we make use of the representation of the solution of the system of equations (11) in the plane domain [4]:

$$\begin{aligned} w_2^{(1)}(x) =& \operatorname{grad}\Phi_1 + \mathrm{r}^2 \operatorname{grad}\left\{\left[\left(\alpha_1 + \tfrac{1}{2}\right)\mathrm{r}\,\tfrac{\partial}{\partial \mathrm{r}} + 2\alpha_1\right]\Phi_2 + \beta_1\left(\mathrm{r}\,\tfrac{\partial}{\partial \mathrm{r}} + 2\right)\Phi_3\right\} - \\ & - xr\,\tfrac{\partial}{\partial r}\left[(2\alpha_1 - 1)\Phi_2 + 2\beta_1\Phi_3\right] + A_0x + B_0\widetilde{x}, \\ w_2^{(2)}(x) =& \operatorname{grad}\Phi_4 + \mathrm{r}^2 \operatorname{grad}\left\{\alpha_2\left(\mathrm{r}\,\tfrac{\partial}{\partial \mathrm{r}} + 2\right)\Phi_2 + \left[\left(\beta_2 + \tfrac{1}{2}\right)\mathrm{r}\,\tfrac{\partial}{\partial \mathrm{r}} + 2\beta_2\right]\Phi_3\right\} - \\ & - xr\,\tfrac{\partial}{\partial r}\left[2\alpha_2\Phi_2 + (2\beta_2 - 1)\Phi_3\right] + C_0x + D_0\widetilde{x}, \end{aligned} \tag{16}$$

where $\alpha_1 = \frac{1}{2\Delta_1}\,(cd - b_1a_2 - \Delta_1)$, $\beta_1 = \frac{1}{2\Delta_1}\,(cb - a_2d)$, $\Delta_1 = a_1a_2 - c^2$, $\alpha_2 = \frac{1}{2\Delta_1}\,(cb - a_1d)$, $\beta_2 = \frac{1}{2\Delta_1}\,(cd - a_1b_2 - \Delta_1)$, $\widetilde{x} = (-x_2, x_1)$, $\Phi_k(x)$ are arbitrary harmonic functions, $k = 1, 2, 3, 4$, $r\,\frac{\partial}{\partial r} = x_1\,\frac{\partial}{\partial x_1} + x_2\,\frac{\partial}{\partial x_2}$.

We rewrite the boundary condition (12) in the normal and tangential components as follows:

$$\{w_2^{(i)}\}_n = f_n^{(i)}, \quad \{w_2^{(i)}\}_s = f_s^{(i)}, \tag{17}$$

where $f_n^{(i)} = n_1f_1^{(i)} + n_2f_2^{(i)}$, $f_s^{(i)} = -n_2f_1^{(i)} + n_1f_2^{(i)}$, $i = 1, 2$, $n = (n_1, n_2)$ and $s = (-n_2, n_1)$ are, respectively, the normal and the tangent with respect to the circle C.

Harmonic functions Φ_k will be sought in the form

$$\Phi_k(x) = \sum_{m=0}^{\infty} \left(\frac{r}{R}\right)^m \left(X_{mk} \cdot \nu_m(\psi)\right), \quad k = 1, 2, 3, 4, \tag{18}$$

where X_{mk} are unknown two-component vectors, $\nu_m(\psi) = (\cos m\psi, \sin m\psi)$.

Assume that the functions $f^{(i)}(z)$ from (12) satisfy all the conditions for their Fourier series expansion

$$f_n^{(i)} = \frac{\alpha_0^{(i)}}{2} + \sum_{m=1}^{\infty}\left(\alpha_m^{(i)} \cdot \nu_m(\psi)\right), \quad f_s^{(i)} = \frac{\beta_0^{(i)}}{2} + \sum_{m=1}^{\infty}\left(\beta_m^{(i)} \cdot s_m(\psi)\right), \tag{19}$$

where $\alpha_m^{(i)}$ and $\beta_m^{(i)}$ are the Fourier coefficients.

We substitute (18) into (16). From the condition (17), taking into account (19) and passing to the limit as $r \to R$, for the definition of unknown vectors X_{mk} we obtain for every m the system of linear equations whose determinant is equal to $D_m = A_1m + A_2$, where $A_1 = -2\alpha_2\beta_1$, $A_2 = (2\alpha_1 - 1)(1 - 2\beta_2) + 4\alpha_2\beta_1$; $\alpha_1, \alpha_2, \beta_1, \beta_2$ are the given constants, $m = 1, 2, \ldots$. From the uniqueness of the above-formulated problem [3] we can conclude that $D_m \neq 0$. The solution of that system we substitute into (18) and for every Φ_k we obtain the expression in the form of a series. Summarizing this series and reasoning as when deducing formula (14), we obtain [7]

$$\sum_{m=1}^{\infty} \frac{1}{mD_m}\left(\frac{r}{R}\right)^m \cos m(\psi - \theta) = \frac{R^4}{A_1}\int_0^1 \frac{1}{t^{p+1}}\,\ln\frac{R}{R_1}\,dt \equiv M_2(x,y),$$

where $p=\frac{A_2}{A_1}$, $R_1^2=R^2+t^2r^2-2Rrt\cos(\psi-\theta)$.

Now we can write the representations (16) and hence the solution of the problem (11)–(12) in the form of the following integral:

$$(w_1^{(1)},w_2^{(2)})=\int_C \Psi(x,y)F(y)\,d_yC,\quad x\in D, \tag{20}$$

where $\Psi(x,y)=\|a_{pq}\|_{4\times4}$, $F(y)=\|\varphi_p\|_{4\times1}$,

$$\begin{aligned}
a_{l1}=&\Big[\frac{\partial}{\partial x_l}\left(t_1r\frac{\partial}{\partial r}+A_2-\gamma\right)-\frac{r^2}{R^2}\frac{\partial}{\partial x_l}\Big\{\left[\left(\alpha_1+\frac12\right)r\frac{\partial}{\partial r}+2\alpha_1\right]\gamma+\\
&+\beta_1\left(r\frac{\partial}{\partial r}+2\right)2\alpha_2\Big\}-\frac{x_l}{R^2}A_2r\frac{\partial}{\partial r}\Big]M_2(x,y)+\frac{x_l}{2}R^2;\\
a_{l2}=&-a_{l1}+A_2\frac{\partial}{\partial x_l}M_2(x,y)+(x_l+\widetilde{x}_l)\frac{R^2}{2};\\
a_{l3}=&-a_{l4}=2\beta_1\frac{R^2-r^2}{R^2}\frac{\partial}{\partial x_l}\left(r\frac{\partial}{\partial r}+1\right)M_2(x,y),\quad l=1,2;\\
a_{k1}=&-a_{k2}=2\alpha_2\frac{R^2-r^2}{R^2}\frac{\partial}{\partial x_{k-2}}\left(r\frac{\partial}{\partial r}+1\right)M_2(x,y);\\
a_{k3}=&\Big[\frac{\partial}{\partial x_{k-2}}\left(t_2r\frac{\partial}{\partial r}+A_2-\eta\right)-\frac{r^2}{R^2}\frac{\partial}{\partial x_{k-2}}\Big\{2\alpha_2\beta_1\left(r\frac{\partial}{\partial r}+2\right)+\\
&+\left[\left(\beta_2+\tfrac12\right)r\tfrac{\partial}{\partial r}+2\beta_2\right]\eta\Big\}+\tfrac{x_{k-2}}{R^2}A_2r\frac{\partial}{\partial r}\Big]M_2(x,y)+\frac{x_{k-2}}{2}R^2;\\
a_{k4}=&-a_{k3}+A_2\frac{\partial}{\partial x_{k-2}}M_2(x,y)+\frac{x_{k-2}+\widetilde{x}_{k-2}}{2}R^2,\quad k=3,4;
\end{aligned}$$

$\eta=1-2\alpha_1$, $\gamma=1-2\beta_2$, $t_1=\alpha_1\gamma+\frac{\gamma}{2}+2\alpha_2\beta_1$, $t_2=\beta_2\eta+\frac{\eta}{2}+2\alpha_2\beta_1$; $F=(\varphi_1,\varphi_2,\varphi_3,\varphi_4)$, $\varphi_1=f_n^{(1)}$, $\varphi_2=f_s^{(1)}$, $\varphi_3=f_n^{(2)}$, $\varphi_4=f_s^{(2)}$. The functions $f_n^{(i)}$ and $f_s^{(i)}$ are defined in (17), $i=1,2$, $d_yC=R\,d\theta$.

Finally, the solution of the above-posed problem can be represented as follows:

$$U(x)=\int_C K(x,y)\Phi(y)\,d_yC+K_0,$$

where $K(x,y)=\|K_{kj}\|_{5\times5}$, $M_2(x,y)=\|a_{pq}\|_{4\times4}$, $\Phi(y)=(f_3(y),F(y))$,

$$K_{kj}(x,y)=(1-\delta_{5k})\delta_{1j}\frac{(R^2-r^2)e_\alpha}{\pi b}\frac{\partial}{\partial x_\beta}M_1(x,y)+(1-\delta_{5k})(1-\delta_{1j})a_{kj}+\delta_{5k}\delta_{1j}\frac{1}{\pi}\ln\frac{1}{\rho},$$

δ_{kj} is the Kronecker symbol, $k,j=1,2,3,4,5$, $\alpha=\begin{cases}1 \text{ if } j=1,2;\\ 2 \text{ if } j=3,4;\end{cases}$ $\beta=\begin{cases}1 \text{ if } j=1,3;\\ 2 \text{ if } j=2,4.\end{cases}$

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Received 23. IV. 2005; accepted 5. IX. 2005.