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AN APPLICATION OF THE BOUNDARY ELEMENT METHOD IN NUMERICAL ANALYSIS OF STRESS CONCENTRATION FOR ELASTIC BODY

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The boundary value problem, which is the mathematical model of stress concentration in elliptical cylinder with elliptic hole is considered. In this paper the homogeneous, isotropic body in the plane deformable state is considered.

Because the cylinder is in the plane deformable state is considered two-dimensional problem in the domain bounded by coordinate lines of elliptical coordinates system θ, α $(0 \leq \theta < \infty,\ 0 \leq \alpha < 2\pi)$ [1]. Let x, y be Cartesian coordinates, then

$$x = c \operatorname{ch} \theta \cos \alpha, \quad y = c \operatorname{sh} \theta \sin \alpha$$

where $c = const$ is a scale coefficient (in our case take c=1). Namely the boundary value problem in stresses for elliptical ring is considered $(\theta_1 < \theta < \theta_2, 0 < \alpha < 2\pi)$.

Statement of the problem. The boundary value problem owing of symmetry we have to set for quarter of the elliptical ring. Find the solution of equilibrium equations system (without of volume forces)[2] in the domain $\Omega = \left\{\theta_1 < \theta < \theta_2,\ 0 < \alpha < \dfrac{\pi}{2}\right\}$, which satisfies following boundary conditions

$$\begin{aligned} &\text{for } \ \theta = \theta_1 : \quad \sigma_{\theta\theta} = 0,\ \sigma_{\theta\alpha} = 0, \\ &\text{for } \ \theta = \theta_2 : \quad \sigma_{\theta\theta} = \mathrm{p},\ \sigma_{\theta\alpha} = 0, \end{aligned} \tag{1}$$

$$\text{for } \alpha = 0,\ \alpha = \frac{\pi}{2} : \quad \mathrm{v} = 0,\ \sigma_{\theta\alpha} = 0, \tag{2}$$

where $\sigma_{\alpha\alpha}$, $\sigma_{\theta\theta}$, $\sigma_{\theta\alpha}$ are components of the stress tensor in system of elliptical coordinates; u, v are components of the vector of displacement at tangents to the coordinate lines α, θ;.

Numerical solution of this problem is obtained by boundary element method, namely method of fictitious load [2].

Numerical procedure. The boundary of considered domain is divided by elements. The numerical solution of considered problem is constructed on the basis of preliminary obtained solution of simple singular problem which is satisfied boundary conditions on all elements of contour. Because every singular solution satisfies partial differential equations in full domain thus in this case is not necessary to cover entire domain by net.

We take advantage of the singular solution [2] of Kelvin problem in case of plane deformable state. The problem acting point force in the point of infinite elastic domain known as Kelvin problem [3].

For carry out numerical procedure of solution of boundary value problem the boundary $\theta = \theta_1$ will divide by elements N_1 and number from 1 to N_1 counter-clockwise, the boundary $\theta = \theta_2$ will divide by elements N_2 and number from $N_1 + 1$ to $N = N_1 + N_2$ clockwise. If the boundary elements are small then they sufficiently exactly describe the boundary of the domain. In this case may be consider that the normal stress $\sigma_n = \sigma_{\theta,\theta} = p$ act on all length of each element of boundary $\theta = \theta_2$, and the tangential stress $\sigma_s = \sigma_{\theta\alpha} = 0$ and each element of boundary $\theta = \theta_1$ are free from tangential and normal stresses. Thus the boundary conditions (1) have following form

$$\begin{aligned} &\text{for} \quad \theta = \theta_1 : \quad \sigma^{\mathrm{i}}_{\theta\theta} = 0, \ \sigma^{\mathrm{i}}_{\theta\alpha} = 0, \quad \mathrm{i} = 1, \ldots, \mathrm{N}_1, \\ &\text{for} \quad \theta = \theta_2 : \quad \sigma^{\mathrm{i}}_{\theta\theta} = \mathrm{p}, \ \sigma^{\mathrm{i}}_{\theta\alpha} = 0, \quad \mathrm{i} = \mathrm{N}_1 + 1, \ldots, \mathrm{N}, \end{aligned} \tag{3}$$

where i is a number of boundary element.

Each element are corresponded on it continuously distribution point forces, for example element j will correspond continuously distribution constant stresses P_s^j and P_n^j on this element. These are not real tangential and normal stresses acting on element j (if stresses act on other element). That is why it is necessary to differ two groups of stresses for all element. For example, for element j exists applied stresses P_s^j and P_n^j, and real stresses σ_s^j σ_n^j which is arise from acting of applied stresses in each element of boundary.

Using solution of Kelvin problem [2] for plane deformation and transformation formula [4] may be count real stresses $\sigma_s^i \equiv \sigma_{\theta\alpha}^i$ and $\sigma_n^i \equiv \sigma_{\theta\theta}^i$ in the middle point $i = 1, \ldots, N$ of each element. Thus we obtain following relation

$$\sigma_s^i = \sum_{j=1}^{N} \left(A_{ss}^{ij} P_s^j + A_{sn}^{ij} P_n^j \right), \quad \sigma_n^i = \sum_{j=1}^{N} \left(A_{ns}^{ij} P_s^j + A_{nn}^{ij} P_n^j \right), \tag{4}$$

where A_{ss}^{ij}, A_{sn}^{ij}, A_{ns}^{ij}, A_{nn}^{ij} are influence boundary coefficients of stresses for considered problem. For example coefficient A_{ns}^{ij} is got real normal stress (σ_n^i) in center of element i which is arise from unit tangential load ($P_s^j = 1$) applied on segment j.

Because elliptical ring has two axes is of symmetry, the boundary value problem is posed for quarter of the ring thus will take into account the existence of symmetric axes. To attain this, we have to insert influence of symmetrical elements to axes ox, oy in influence coefficients of stresses of boundary elements.

P_s^j and P_n^j are chosen as that (4) satisfies the boundary condition (3). Obtain $2N$ linear algebraic equations with $2N$ unknowns

$$\left\{ \begin{aligned} &\sum_{j=1}^{N} (A_{ss}^{ij} P_s^j + A_{sn}^{ij} P_n^j) = 0, \quad i = 1, \ldots, N, \\ &\sum_{j=1}^{N} (A_{ns}^{ij} P_s^j + A_{nn}^{ij} P_n^j) = \left\{ \begin{array}{l} 0, \ i = 1, \ldots, N_1, \\ p, \ i = N_1 + 1, \ldots, N, \end{array} \right. \end{aligned} \right. \tag{5}$$

where P_s^j and P_n^j are fictitious quantity, which have not physical meaning but the linear combination (4) have one, on this basis compile system (5).

After solve equations system (5) using arbitrary method of numerical analysis may be express the displacement and stresses in each point of body. In particular, numerical value of stress tensor is obtained in some proper points of domain Ω.

Numerical results and observation. The numerical count of problem (1), (2) fulfill in system MATLAB on the personal computer and constract corresponding graph. In computing program symmetry conditions (2) are foresee. The boundaries of quarter of elliptic ring ($\theta = \theta_l$, $l = 1, 2$) are divide into 30 elements i. e. $N_1 = N_2 = 30$, $N = 60$. $\theta_2 = 3$, $0 < \theta_1 < \theta_2$, $\nu = 0.3$, $E = 7 \cdot 10^4$ $p = 1$ (ν is a Poisson coefficient, E is a a the elasticity module).

The plots representing redistribution of the stress $N_{\alpha,\alpha}$ in mentioned points are constructed for different values of $\theta = \theta_1$.

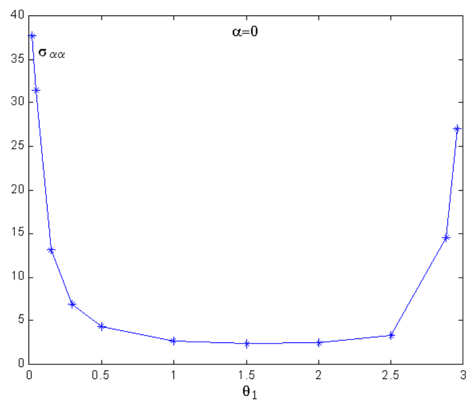

Fig.1.

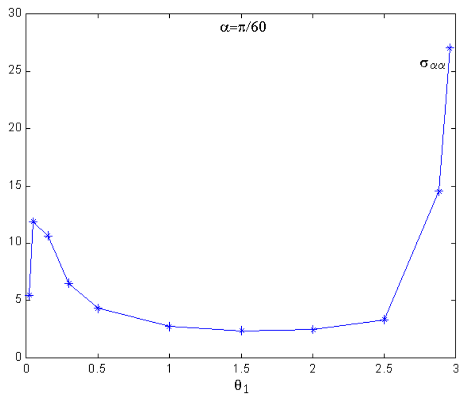

Fig.2

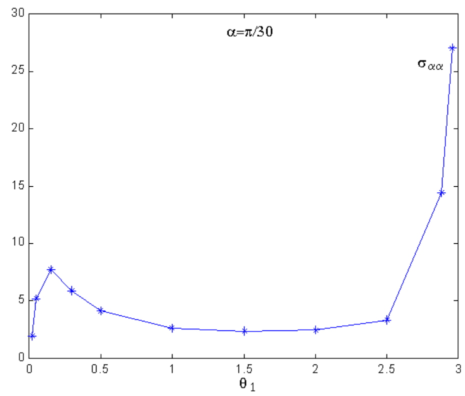

Fig.3.

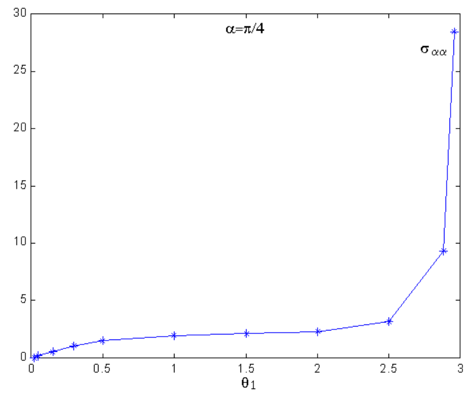

Fig.4

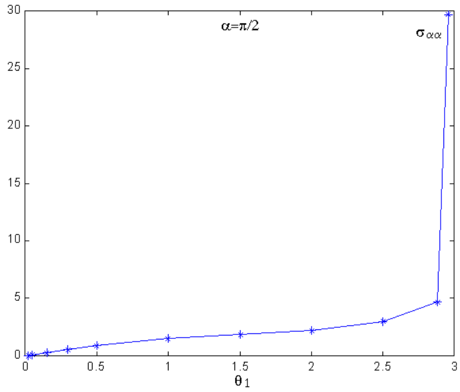

Fig.5.

As it is possible from fig.1 it is evident, which for fixed $\alpha = 0$ and variation of θ_1 $N_{\alpha\alpha} \to \infty$, when $\theta_1 \to 0$ or $\theta_1 \to \theta_2$. When $\theta_1 \to 0$ to take place stress concentration, if $\theta_1 \to \theta_2$ thickness of the ring is vanish, it provoke infinitely increase the stress $N_{\alpha\alpha}$.

From fig.2 and fig.3 it is evident, which $\alpha = \pi/60$ or $\alpha = \pi/30$ and variation of the θ_1 curvature of ellipse decrease when $\theta_1 \to 0$, and stress $N_{\alpha\alpha}$ is small. If the curvature have maximum value the stress is maximum and then decrease. As the same case if $\theta_1 \to \theta_2$ then $N_{\alpha\alpha} \to \infty$.

In case of fig.4 and fig.5, we obtain the same results as in previous case, except the difference that the stress has not maximum.

R E F E R E N C E S

1. Bermant A.F. Mapping. Linear Coordinates. Transformation. Green's Formulas. Moscow, 1958 (Russian).

2. Crouch S.L., Starfield A.M. Boundary element methods in solid mechanics, GEORGE ALLEN & UNWIN 1983, London-Boston-Sydney.

3. Sokolnikoff I.S. Mathematical theory of elasticity, 1956, 2nd edn. -New York: McGraw-Hill, 1956.

4. Muskhelishvili N.I. Some Basic Problems of the Mathematical Theory of Elasticity, Noordhoff, Groningen, Holland, 1953.

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