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THE EXAMPLES OF NON-INTEGER VERTICES OF THE RELAXATION POLYTOPE OF THE LINEAR ORDERING PROBLEM AND THEIR STRUCTURE

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Abstract

For the the NP-hard linear ordering problem construct about 10 classes of facets [1,...,6], where each class contains exponential number facets. Therefore our aim can be formulated as follows: by solving linear ordering problem step-by-step we construct needed facets by a polynomial algorithm. Consequently, is studied in this article the examples of non-integer vertices of the relaxation polytope of the linear ordering problem.

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Let's consider $K_n = (V_n, A)$ a complete directed graph $V_n = \{1, ..., n\}$ and weights c_{ij} for each edge $(i, j) \in A$. The set of acyclic tournaments is denoted by T_n. The linear ordering problem can be formulated in the following way: to find acyclic tournament maximal weight in complete edge weight directed graph.

Incidence points $x^T \in R^{n^2-n}$ correspond to each acyclic tournament $T \subset T_n$ in the following way:

$$x_{ij}^T = \begin{cases} 1, & (i,j) \in T, \\ 0, & (i,j) \notin T. \end{cases}$$

Let's consider now the problem:

$$\max \sum_{i=1,\, i \neq j}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\begin{array}{l} 0 \le x_{ij} \le 1, \\ x_{ij} + x_{ji} = 1, \\ 0 \le x_{ij} + x_{jk} - x_{ik} \le 1, \\ i \neq j, i \neq k, j \neq k, 1 \le i, j, k \le n. \end{array} \tag{1}$$

We have denoted the relaxational polytope (1) of the linear ordering problem by B_n. Polytope B_n has integer vertices corresponding one-to-one to the admissible solutions of the linear ordering problem as well as the non-integer vertices.

Using the equalities $x_{ij} + x_{ji} = 1, i \neq j, 1 \le i, j \le n$; it is easy to show that the system of equalities (1) can be rewriting in the following form

$$\begin{array}{l} 0 \le x_{ij}, \\ x_{ij} + x_{ji} = 1, \\ 0 \le x_{ij} + x_{jk} - x_{ik}, \\ i \neq j, i \neq k, j \neq k, 1 \le i, j, k \le n. \end{array} \tag{2}$$

The system of equalities $x_{ij}+x_{ji}=1, i\neq j, 1\leq i,j\leq n$, allow us to consider the linear ordering problem in the $(n^2-n)/2$ dimensional space.

Lemma 1.

$$x_{ij}+x_{jk}-x_{ik}=0\Rightarrow x_{jk}+x_{ki}-x_{ji}=0, x_{ki}+x_{ij}-x_{kj}=0,$$

$$x_{ji}+x_{ik}-x_{jk}=1, x_{ik}+x_{kj}-x_{ij}=1, x_{kj}+x_{ji}-x_{ki}=1.$$

If we take into consideration the system of equalities $x_{ij}+x_{ji}=1, i\neq j, 1\leq i,j\leq n$, the proof is evident.

Corollary 1. Out of lemma 1 it appears that having of any equality out of the six we can obtain the rest of them.

Lemma 2.

$$1. x_{ij}+x_{jk}-x_{ik}=0, x_{ik}+x_{kl}-x_{il}=0\Leftrightarrow x_{ij}+x_{jl}-x_{il}=0, x_{jk}+x_{kl}-x_{jl}=0,$$

$$2. x_{ij}+x_{jk}-x_{ik}=1, x_{ik}+x_{kl}-x_{il}=1\Leftrightarrow x_{ij}+x_{jl}-x_{il}=1, x_{jk}+x_{kl}-x_{jl}=1,$$

$$3. x_{ij}=0, x_{jk}=0\Leftrightarrow x_{ik}=0, x_{ij}+x_{jk}-x_{ik}=0,$$

$$4. x_{ij}=1, x_{jk}=1\Leftrightarrow x_{ik}=1, x_{ij}+x_{jk}-x_{ik}=1,$$

proof.

$$x_{ij}+x_{jk}-x_{ik}+x_{ik}+x_{kl}-x_{il}=x_{ij}+x_{jk}+x_{kl}-x_{il}=x_{ij}+x_{jl}-x_{il}+x_{jk}+x_{kl}-x_{jl}$$

Taking into account that $x_{ij}+x_{jk}-x_{ik}\geq 0, x_{ik}+x_{kl}-x_{il}\geq 0, x_{ij}+x_{jl}-x_{il}\geq 0, x_{jk}+x_{kl}-x_{jl}\geq 0$, we may obtain the proof of the first variant. Substantiation of the other variants can be obtained in analogous way.

Let $x_{in+1}=0, 1\leq i\leq n$, then equality $x_{ij}=0$ can be rewrited in the form $x_{ij}+x_{jn+1}-x_{in+1}=0$. Now let's consider the set M as such as if $x_{ij}+x_{jk}-x_{ik}=0$ then $(i,j,k)\in M$. Consequently the first and the third variants of lemma 2 can be rewrited in the form of

$$(i,j,k)\in M, (i,k,l)\in M\Leftrightarrow (i,j,l)\in M, (j,k,l)\in M,$$

$$(i,j,n+1)\in M, (j,k,n+1)\in M\Leftrightarrow (i,k,n+1)\in M, (i,j,k)\in M.$$

Definition 1. If x^0 is a non-integer vertex of the polytope B_n on the set I and for each equality of the system of equalities that give solution x^0, exists such basis, that in this basis an equality is changed to the opposite value we obtain adjacent integer vertex, then x^0 will be called facet non-integer vertex at the set I.

Example 1. We consider non-integer vertex x^0 of the polytope B_n, at the set $\{1,...,8\}$, where denominators of the non-integer coordinates equal to 3.

$$x_{12}=0, x_{13}=0, x_{24}=1, x_{26}=1, x_{34}=1, x_{37}=1, x_{46}=0, x_{48}=0,$$

$$x_{75}=0, x_{78}=0, x_{68}=0, x_{58}=0, x_{17}=2/3, x_{13}=1/3, x_{18}=1/3, x_{16}=1/3,$$

$$x_{28}=1/3, x_{14}=1/3, x_{27}=2/3, x_{36}=1/3, x_{25}=1/3, x_{23}=2/3, x_{38}=1/3,$$

$$x_{35}=2/3, x_{45}=1/3, x_{45}=1/3, x_{76}=1/3, x_{65}=1/3.$$

The point x^0 satisfies the following system of equalities:

$$\begin{array}{l}
x_{12}=0, x_{13}=0, x_{43}=0, x_{73}=0, \\
(1,2,7)\in M, (1,4,7)\in M, (4,7,5)\in M, \\
(2,7,3)\in M, (2,3,6)\in M \Leftrightarrow (2,7,6)\in M, (7,3,6)\in M; \\
(2,5,3)\in M, (2,3,6)\in M \Leftrightarrow (2,5,6)\in M, (5,3,6)\in M; \\
(2,5,4)\in M, (2,4,6)\in M \Leftrightarrow (2,5,6)\in M, (5,4,6)\in M; \\
(1,3,6)\in M, (1,6,8)\in M \Leftrightarrow (1,3,8)\in M, (3,6,8)\in M; \\
(1,4,6)\in M, (1,6,8)\in M \Leftrightarrow (1,4,8)\in M, (4,6,8)\in M; \\
(2,5,8)\in M, (1,2,8)\in M \Leftrightarrow (1,2,5)\in M, (1,5,8)\in M; \\
x_{46}=0, x_{62}=0 \Leftrightarrow x_{42}=0, (4,6,2)\in M; \\
x_{46}=0, x_{68}=0 \Leftrightarrow x_{48}=0, (4,6,8)\in M; \\
x_{75}=0, x_{58}=0 \Leftrightarrow x_{78}=0, (7,5,8)\in M;
\end{array} \tag{3}$$

This system contains the following non-basic system of equalities (examples 2, 3, 4) :

Example 2.

$$\begin{array}{l}
x_{12}=0, x_{13}=0, x_{62}=0, x_{68}=0, x_{58}=0, \\
(2,5,3)\in M, (2,3,6)\in M \Leftrightarrow (2,5,6)\in M, (5,3,6)\in M; \\
(1,3,6)\in M, (1,6,8)\in M \Leftrightarrow (1,3,8)\in M, (3,6,8)\in M; \\
(2,5,8)\in M, (1,2,8)\in M \Leftrightarrow (1,2,5)\in M, (1,5,8)\in M;
\end{array} \tag{4}$$

Equality $x_{73}=0$ is necessary to be added to this non-basis system of equalities in order to obtain the facet non-integer vertex of the polytope B_n at the set $\{1,6,5,2,3,8\}$.

Example 3.

$$x_{12}=0, x_{13}=0, x_{73}=0, x_{78}=0, x_{68}=0, x_{62}=0,$$

$$(2,7,3)\in M, (2,3,6)\in M \Leftrightarrow (2,7,6)\in M, (7,3,6)\in M;$$

$$(1,3,6)\in M, (1,6,8)\in M \Leftrightarrow (1,3,8)\in M, (3,6,8)\in M;$$

$$(1,2,8)\in M, (1,2,7)\in M.$$

If we add either of equalities $(1,7,8)\in M$ and $(2,7,8)\in M$, the system of equalities corresponding to facet non-integer vertex of the polytope B_n at the set {1,7,6,2,3,8} will be obtained.

Example 4.

$$\begin{array}{l}
x_{12}=0, x_{13}=0, x_{43}=0, x_{73}=0, \\
(1,2,7)\in M, (1,4,7)\in M, (1,3,6)\in M, (1,4,6)\in M, \\
(2,7,3)\in M, (2,3,6)\in M \Leftrightarrow (2,7,6)\in M, (7,3,6)\in M; \\
x_{46}=0, x_{62}=0 \Leftrightarrow x_{42}=0, (4,6,2)\in M;
\end{array} \tag{5}$$

If we add either of equalities $X_{47}=0$ and $(4,7,3)\in M$, we obtain facet non-integer vertex of the polytope B_n, at the set {1,2,3,4,6,7}.

In the following examples 5, 6 we consider the non-integer vertices which are adjacent of the non-integer vertex x^0 from am example 1.

Example 5.

$$x_{12}=0, x_{13}=0, x_{43}=0, (1,2,7)\in M, (4,7,5)\in M;$$

$$(2,7,3)\in M, (2,3,6)\in M \Leftrightarrow (2,7,6)\in M, (7,3,6)\in M;$$

$$(2,5,3)\in M, (2,3,6)\in M \Leftrightarrow (2,5,6)\in M, (5,3,6)\in M;$$

$$(2,5,4)\in M, (2,4,6)\in M \Leftrightarrow (2,5,6)\in M, (5,4,6)\in M;$$

$$(1,3,6)\in M, (1,6,8)\in M \Leftrightarrow (1,3,8)\in M, (3,6,8)\in M;$$

$$(1,4,6)\in M, (1,6,8)\in M \Leftrightarrow (1,4,8)\in M, (4,6,8)\in M;$$

$$(2,5,8)\in M, (1,2,8)\in M \Leftrightarrow (1,2,5)\in M, (1,5,8)\in M;$$

$$x_{46}=0, x_{62}=0 \Leftrightarrow x_{42}=0, (4,6,2)\in M;$$

$$x_{46}=0, x_{68}=0 \Leftrightarrow x_{48}=0, (4,6,8)\in M;$$

$$x_{75}=0, x_{58}=0 \Leftrightarrow x_{78}=0, (7,5,8)\in M;$$

$$x_{75}=0, x_{53}=0 \Leftrightarrow x_{73}=0, (7,5,3)\in M;$$

where non-integer coordinates are equal to 1/2. This non-integer vertex is not the facet non-integer vertex of the polytope B_n, however, it contains facet non-integer vertex obtained in example 2.

Example 6.

$$x_{12}=0, x_{13}=0, x_{43}=0, x_{73}=0, x_{35}=0,$$

$$(1,2,7)\in M, (1,4,7)\in M, (4,7,5)\in M,$$

$$(2,7,3)\in M, (2,3,6)\in M \Leftrightarrow (2,7,6)\in M, (7,3,6)\in M;$$

$$(2,5,4)\in M, (2,4,6)\in M \Leftrightarrow (2,5,6)\in M, (5,4,6)\in M;$$

$$(1,3,6)\in M, (1,6,8)\in M \Leftrightarrow (1,3,8)\in M, (3,6,8)\in M;$$

$$(1,4,6)\in M, (1,6,8)\in M \Leftrightarrow (1,4,8)\in M, (4,6,8)\in M;$$

$$(2,5,8)\in M, (1,2,8)\in M \Leftrightarrow (1,2,5)\in M, (1,5,8)\in M;$$

$$x_{46}=0, x_{62}=0 \Leftrightarrow x_{42}=0, (4,6,2)\in M;$$

$$x_{46}=0, x_{68}=0 \Leftrightarrow x_{48}=0, (4,6,8)\in M;$$

$$x_{75}=0, x_{58}=0 \Leftrightarrow x_{78}=0, (7,5,8)\in M.$$

We obtain that

$$x_{15}=0, x_{25}=0, x_{45}=0, x_{65}=0, x_{18}=0, x_{28}=0, x_{38}=0, x_{48}=0, x_{47}=0.$$

Consequently we have facet non-integer vertex at the set {1,2,3,4,6,7} obtained in example 4.

REFERENCES

1. Grotschel M., Junger M., Reinelt G. Facets of the linear ordering polytope, Math. Progr. 33, 43-60 (1985).

2. Reinelt G. The linear ordering problem: algorithms and applications, Heldermann. Verlag. Berlin (1986).

3. Bolotashvili G. On the facets of the permutation polytope, Georgian Academy of Sciences 121 (2) 281-284 (1986).

4. Bolotashvili G. A class of facets of the permutation polytope and a method for constructing facets of the permutation polytope, VINITI 3403-B87, 3405-B87 (1987).

5. Leung J., Lee J. More facets from fences for linear ordering and acyclic subgraph polytope, Discrete Applied Math. 50, 185-200 (1992).

6. Bolotashvili G., Kovalev M., Girlich E. New facets of the linear ordering polytope, SIAM J. Discrete Mathematics 12(3) 326-336 (1999).

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