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SOLUTION OF OUTSIDE PROBLEM OF PLANE ELASTICITY THEORY BY BOUNDARY ELEMENT METHOD

Zirakashvili N.

I. Vekua Institute of Applied Mathematics

In the present paper is considered the boundary value problem for infinite domain with semi-moon like cut. The all-around tension is given at the infinity and the contour of cut is free from stresses.

The inner boundary of this domain may be describe by coordinate lines of bipolar coordinates system θ, α $(-\infty < \theta < \infty, 0 \leq \alpha < 2\pi)$ [1]. Let x, y be Cartesian coordinates, then

$$x = \frac{\operatorname{sh}\theta}{\operatorname{ch}\theta + \cos\alpha}, \quad y = \frac{\sin\alpha}{\operatorname{ch}\theta + \cos\alpha},$$

where the coordinate lines are two orthogonal circles system, their centers are on the axes ox and oy accordingly.

Thus inner bound of considered domain is described by lines $\alpha = \alpha_1$ and $\alpha = \alpha_2$. Because domain is symmetrical to axis y, we have to form the boundary value problem in half-domain. This means we have to find the solution of system of equilibrium equations (without volume forces) [2] in the domain $\Omega = \{0 < \theta < \infty,\ \alpha_1 < \alpha < \alpha_2\}$, with satisfies following boundary conditions

$$\text{for } \ \alpha = \alpha_1: \quad \sigma_{\alpha\alpha} = 0,\ \sigma_{\theta\alpha} = 0, \tag{1}$$

$$\text{for } \ \alpha = \alpha_1: \quad \sigma_{\alpha\alpha} = 0,\ \sigma_{\theta\alpha} = 0, \tag{2}$$

$$\text{for } \ \theta \to \infty: \quad \sigma_{\theta\theta} = \mathrm{p},\ \sigma_{\theta\alpha} = 0, \tag{3}$$

$$\text{for } \ \theta = 0: \quad \mathrm{v} = 0,\ \sigma_{\theta\alpha} = 0,$$

where $\sigma_{\alpha\alpha}$, $\sigma_{\theta\theta}$, $\sigma_{\theta\alpha}$ are components of the stress tensor in system of bipolar coordinates; u, v are components of the vector of displacement at tangents to the coordinate lines θ, α.

The solution of the boundary value problem is received of boundary element method, namely by fictitious load method [2].

The boundary of considered domain is divided by elements. The numerical solution of considered problem is constructed on the basis of preliminary obtained solution of simple singular problem which is satisfied boundary conditions on all elements of contour. Because every singular solution satisfies partial differential equations in full domain thus in this case is not necessary to cover entire domain by net.

In this case we will use Kelvin problem [3], i.e. the singular solutin of boundary value problem such that acting point force in the point of infinite elastic domain.

To carry out the numerical procedure of the boundary value problem (1), (2), (3) the boundary $\alpha = \alpha_2$ divide by elements N_1 and number from 1 to N_1 counter-clockwise and $\alpha = \alpha_1$ boundary divide by elements N_2 and number from N_1+1 to $N = N_1+N_2$ counter-clockwise.

In order to solve outer problem when are given non-zero stresses at infinity it is necessary to satisfy boundary condition which obtain by acting stresses on the boundary of cut taking by sign ,,minus". Full solution obtains by add of additional (induced) and ,,origin" stresses. Also ,,origin" stresses are equal to stresses given at infinity. Finally summary stresses (normal and tangential stresses) on the contour of cut will be equal zero.

Thus the boundary conditions (2), (3) in additional stresses will written by following form:

$$\text{for}:\ \alpha=\alpha_1\ \ \alpha=\alpha_2:\quad \sigma_{\rm s}\equiv\sigma_{\theta\alpha}=-\tau_{\rm s},\quad \sigma_{\rm n}\equiv\sigma_{\alpha\alpha}=-\tau_{\rm n}, \tag{4}$$

where τ_s, τ_n are additional tangential and normal stresses, which exspress by σ_{xx}, σ_{yy}, σ_{xy}.

If boundary elements are small they sufficiently exactly describe the boundary of domain. In this case we can suppose that each element on full length of boundary $\alpha = \alpha_1$ and $\alpha = \alpha_2$ act constant normal and tangential stresses $\sigma_{\alpha\alpha} = -\tau_n$, $\sigma_{\theta\alpha} = -\tau_s$ accordingly. Thus the boundary conditions (4) will have following form

$$\begin{aligned}&\text{for } \alpha=\alpha_2:\ \sigma^{\rm i}_{\theta\alpha}=-\tau^{\rm i}_{\rm s},\ \sigma^{\rm i}_{\alpha\alpha}=-\tau^{\rm i}_{\rm n},\quad {\rm i}=1,\ldots,{\rm N}_1,\\ &\text{for } \alpha=\alpha_1:\ \ \sigma^{\rm i}_{\theta\alpha}=-\tau^{\rm i}_{\rm s},\ \sigma^{\rm i}_{\alpha\alpha}=-\tau^{\rm i}_{\rm n},\quad {\rm i}={\rm N}_1+1,\ldots,{\rm N},\end{aligned} \tag{5}$$

where i is a number of the boundary element.

$$\tau^i_s = (\sigma_{yy}-\sigma_{xx})\cos\beta^i\sin\beta^i - \sigma_{xy}(\cos^2\beta^i-\sin^2\beta^i) = 0,$$

$$\tau^i_n = \sigma_{xx}\sin^2\beta^i - 2\sigma_{xy}\cos\beta^i\sin\beta^i + \sigma_{yy}\cos^2\beta^i = p,$$

β^i is an angle between global and local coordinate system which beginning is in the middle-point of element i.

Every element is corresponded on it continuously distribution point forces, for example element j will correspond continuously distribution constant stresses P^j_s and P^j_n on this element. These are real tangential and normal stresses acting on element j (if stresses act on other elements). That is why it is necessary to differ two groups of stresses for each element. For example exist applied stresses P^j_s and P^j_n for element j and real stresses σ^j_s, σ^j_n which is arise from acting of applied stresses in all element of boundary.

Using solution of Kelvin problem [2] and transformation formula [4] it is possible to count real stresses $\sigma^i_s \equiv \sigma^i_{\theta\alpha}$ and $\sigma^i_n \equiv \sigma^i_{\alpha\alpha}$ in the middle point of each elements $i = 1,\ldots,N$. Thus obtain following relations

$$\sigma^i_s = \sum_{j=1}^{N}\left(A^{ij}_{ss}P^j_s + A^{ij}_{sn}P^j_n\right),\quad \sigma^i_n = \sum_{j=1}^{N}\left(A^{ij}_{ns}P^j_s + A^{ij}_{nn}P^j_n\right), \tag{6}$$

where A^{ij}_{ss}, A^{ij}_{sn}, A^{ij}_{ns}, A^{ij}_{nn} are influence coefficients of stresses of boundary elements for considered problem. For example coefficient A^{ij}_{ns} is giving real normal stress (σ^i_n)

in the center of element i, which is arise from unit tangential load ($P_s^j = 1$) applied on the segment j.

Because considered domain has axis of symmetry the boundary value problem is posed for semi-domain, thus will take into account the existence of symmetric axis. To attain this, we have to insert influence of symmetrical elements to axis y in influence coefficients of stresses of boundary elements.

P_s^j and P_n^j choose as that (6) satisfies the boundary conditions (5). Obtain $2N$ linear algebraic equations with $2N$ unknowns.

$$\begin{cases} \sum\limits_{j=1}^{N} (A_{ss}^{ij} P_s^j + A_{sn}^{ij} P_n^j) = 0, \\ \sum\limits_{j=1}^{N} (A_{ns}^{ij} P_s^j + A_{nn}^{ij} P_n^j) = -p, \ i = 1, \ldots, N, \end{cases} \tag{7}$$

where P_s^j and P_n^j are fictitious quantities. They were introduce as means of numerical solution partial problem and have not physical meaning, but the linear combination (6) have one.

After solve system of equations (7) using arbitrary method of numerical analysis will be count the tangential stresses by fictitious load P_s^j and P_n^j $j = 1, \ldots, N$ on the boundary of cut.

The numerical count of problems (1), (2), (3) fulfill in the system MATLAB on the personal computer and construct corresponding graph. The boundary of cut is divide into 90 elements i.e. $N = 180$. $0 < \theta < 10$, $\nu = 0.3$, $E = 7 \cdot 10^4$ $p = 10$. (ν is a Poisson coefficient, E is the elasticity module).

The computing program for numerical solution of problem (1), (2), (3) have very simple structure. The calculation is fulfilled in general by four separate step. 1) Determination of local boundary elements and give boundary conditions for all elements in displacement or stresses. 2) Calculation the influence boundary coefficients and construction of algebraic equations system and solution with taking into account boundary conditions. 3) Calculation the displacement and stress in each boundary elements. 4) Calculation of influence coefficients for given inner points of considered domain and calculation of displacement and stresses.

The plots representing below redistribution of stress $N_{\theta\theta}$ give on the boundary of cut. On the fig.1 is given tangential stress when $\alpha_1 = \pi/2 - \pi/180$ $\alpha_2 = \pi/2$. On the fig.2 - fig 5. is given tangential stress $N_{\theta\theta}$, when $\alpha_2 = \pi/2$ and α_1 equal $\pi/3$, $\pi/4$, $\pi/6$, $\pi/180$ accordingly i.e. the size of cut is increase. In the angular point $(1;0)$ the stresses $N_{\theta\theta}$ tend to infinity i.e. take place stress concentration.

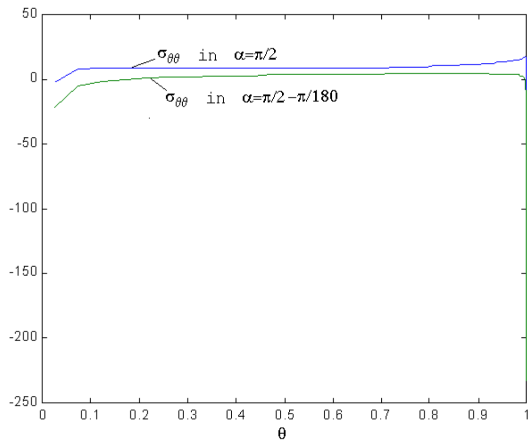

Fig.1.

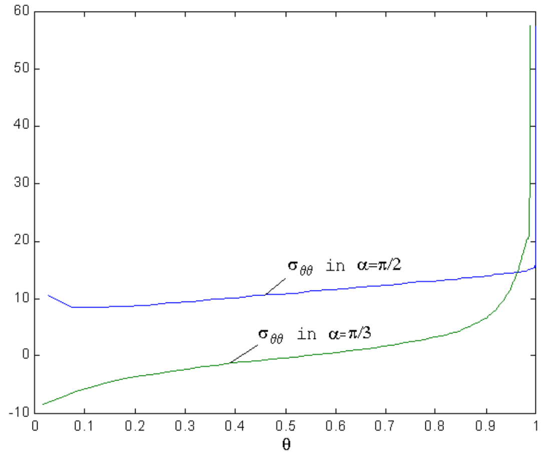

Fig.2

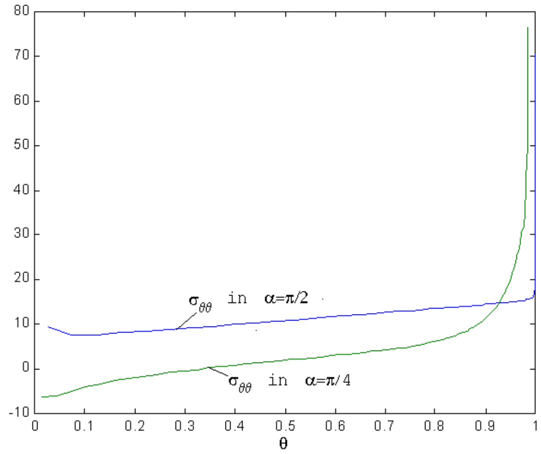

Fig.3.

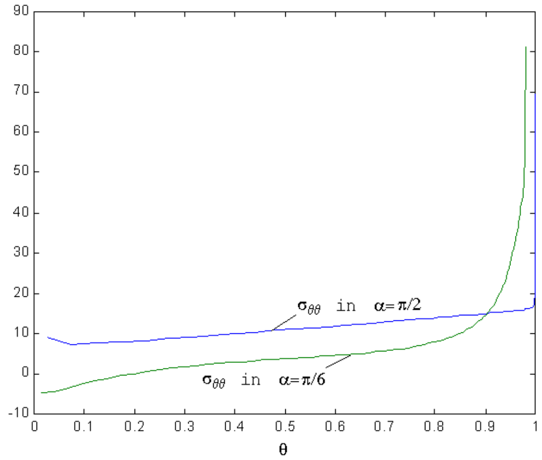

Fig.4

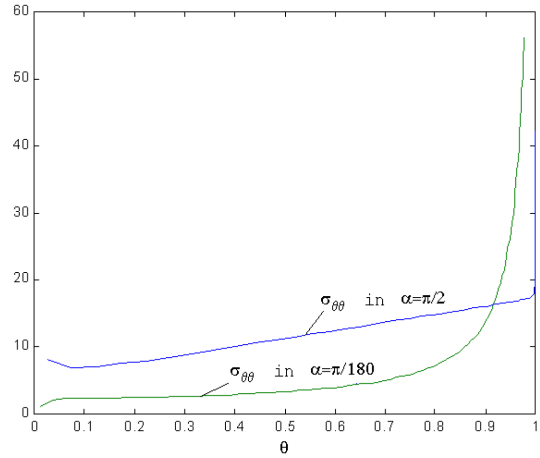

Fig.5.

R E F E R E N C E S

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