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BOUT PLANE THEORY FOR HEMITROPIC ELASTIC MATERIALS

Janjgava R. Gulua B.

I. Vekua Institute of Applied Mathematics
Tbilisi State University

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Equilibrium equations of statics of the hemitropic theory of elasticity have the form [1]

$$\frac{\partial \tau_{ij}}{\partial x_i} + F_j = 0, \quad \frac{\partial \mu_{ij}}{\partial x_i} + \varepsilon_{jpq}\tau_{pq} + G_j = 0, \quad j = 1,2,3,$$

where $\{\tau_{ij}\}$ is the tensor of the force stress, $\{\mu_{ij}\}$ is the tensor of the couple stress, $\vec{F} = (F_1, F_2, F_3)^T$ and $\vec{G} = (G_1, G_2, G_3)^T$ are the body force and body couple vectors, ε_{jpq} is the permutation (Levi-Civita) symbol.

The tensors of the force stress $\{\tau_{ij}\}$ and the couple stress $\{\mu_{ij}\}$ in the linear theory are as follows (the constitutive equations)

$$\tau_{ij} = (\mu+\alpha)\frac{\partial u_j}{\partial x_i} + (\mu-\alpha)\frac{\partial u_i}{\partial x_j} + \lambda\delta_{ij}\frac{\partial u_k}{\partial x_k} + \delta\delta_{ij}\frac{\partial \omega_k}{\partial x_k} + (k+\nu)\frac{\partial \omega_j}{\partial x_i} + (k-\nu)\frac{\partial \omega_i}{\partial x_j} - 2\alpha\varepsilon_{ijk}\omega_k,$$

$$\mu_{ij} = \delta\delta_{ij}\frac{\partial u_k}{\partial x_k} + (k+\nu)\left[\frac{\partial u_j}{\partial x_i} - \varepsilon_{ijk}\omega_k\right] + \beta\delta_{ij}\frac{\partial \omega_k}{\partial x_k} + (k-\nu)\left[\frac{\partial u_i}{\partial x_j} - \varepsilon_{ijk}\omega_k\right] + (\gamma+\varepsilon)\frac{\partial \omega_j}{\partial x_i} + (\gamma-\varepsilon)\frac{\partial \omega_i}{\partial x_j},$$

where $\vec{u} = (u_1, u_2, u_3)^T$ is the displacement vector, $\vec{\omega} = (\omega_1, \omega_2, \omega_3)^T$ is the microrotation vector, δ_{ij} is the Kronecker delta and $\alpha, \beta, \gamma, \delta, \lambda, \mu, \nu, \kappa$ and ε are the material constants.

Let $\vec{u}, \vec{\omega}, \vec{F}$ and $\vec{G}$ are independent of the variable x_3 [2].

Equilibrium equations of the force stress and the couple stress have the following complex forms:

$$\begin{aligned}
&\frac{\partial}{\partial z}\left(\tau_{11} - \tau_{22} + i(\tau_{12} + \tau_{21})\right) + \frac{\partial}{\partial \bar{z}}\left(\tau_{11} + \tau_{22} + i(\tau_{12} - \tau_{21})\right) + F_+ = 0,\\
&\frac{\partial}{\partial z}\tau_+ + \frac{\partial}{\partial \bar{z}}\bar{\tau}_+ + F_3 = 0,\\
&\frac{\partial}{\partial z}\left(\mu_{11} - \mu_{22} + i(\mu_{12} + \mu_{21})\right) + \frac{\partial}{\partial \bar{z}}\left(\mu_{11} + \mu_{22} + i(\mu_{12} - \mu_{21})\right) - i\tau_+ + i_+\tau + G_+ = 0, \qquad (1)\\
&\frac{\partial}{\partial z}\mu_+ + \frac{\partial}{\partial \bar{z}}\bar{\mu}_+ + \tau_{12} - \tau_{21} + G_3 = 0,
\end{aligned}$$

where

$$z = x_1 + ix_2,\quad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right)\quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right),$$
$$F_+ = F_1 + iF_2,\quad G_+ = G_1 + iG_2,\quad \tau_+ = \tau_1 + i\tau_2 .$$

Complex combination of the force stresses and couple stresses are represented as follows

$$\begin{aligned}
&\tau_{11} - \tau_{22} + i(\tau_{12} + \tau_{21}) = 4\mu\frac{\partial u_+}{\partial \bar{z}} + 4k\frac{\partial \omega_+}{\partial \bar{z}},\\
&\tau_{11} + \tau_{22} + i(\tau_{12} - \tau_{21}) = 2(\lambda + \mu + \alpha)\frac{\partial u_+}{\partial z} + 2(\lambda + \mu - \alpha)\frac{\partial \bar{u}_+}{\partial \bar{z}} + 2(\delta + k + \nu)\frac{\partial \omega_+}{\partial z} +\\
&+ 2(\delta + k - \nu)\frac{\partial \bar{\omega}_+}{\partial \bar{z}} - 4\alpha i\omega_3,\\
&\tau_+ = \tau_{13} + i\tau_{23} = 2(\mu + \alpha)\frac{\partial u_3}{\partial \bar{z}} + 2(k + \nu)\frac{\partial \omega_3}{\partial \bar{z}} - 2\alpha i\omega_+,\\
&{}_+\tau = \tau_{31} + i\tau_{32} = 2(\mu - \alpha)\frac{\partial u_3}{\partial \bar{z}} + 2(k - \nu)\frac{\partial \omega_3}{\partial \bar{z}} + 2\alpha i\omega_+,\\
&\mu_{11} - \mu_{22} + i(\mu_{12} + \mu_{21}) = 4\kappa\frac{\partial u_+}{\partial \bar{z}} + 4\gamma\frac{\partial \omega_+}{\partial \bar{z}},\\
&\mu_{11} + \mu_{22} + i(\mu_{12} - \mu_{21}) = 2(\delta + k + \nu)\frac{\partial u_+}{\partial z} + 2(\delta + k - \nu)\frac{\partial \bar{u}_+}{\partial \bar{z}} + 2(\beta + \gamma + \varepsilon)\frac{\partial \omega_+}{\partial z} +\\
&+ 2(\beta + \gamma - \varepsilon)\frac{\partial \bar{\omega}_+}{\partial \bar{z}} - 4\nu i\omega_3.\\
&\mu_+ = \mu_{13} + i\mu_{23} = 2(k + \nu)\frac{\partial u_3}{\partial \bar{z}} + 2(\gamma + \varepsilon)\frac{\partial \omega_3}{\partial \bar{z}} - 2\nu i\omega_+
\end{aligned} \tag{2}$$

By substituting (2) into (1) we obtain equilibrium equations in components of displacement and microrotation vectors:

$$\begin{aligned}
&(\mu + \alpha)\Delta u_+ + (k + \nu)\Delta\omega_+ + 2(\lambda + \mu - \alpha)\frac{\partial \theta_1}{\partial \bar{z}} + 2(\delta + k - \nu)\frac{\partial \theta_2}{\partial \bar{z}} - 4\alpha i\frac{\partial \omega_3}{\partial \bar{z}} + F_+ = 0,\\
&(\mu + \alpha)\Delta u_3 + (k + \nu)\Delta\omega_3 + 2\alpha i\left(\frac{\partial \bar{\omega}_+}{\partial \bar{z}} - \frac{\partial \omega_+}{\partial z}\right) + F_3 = 0,\\
&(k + \nu)\Delta u_+ + (\gamma + \varepsilon)\Delta\omega_+ + 2(\delta + k - \nu)\frac{\partial \theta_1}{\partial \bar{z}} + 2(\beta + \gamma - \varepsilon)\frac{\partial \theta_2}{\partial \bar{z}} - 4\alpha i\frac{\partial u_3}{\partial \bar{z}} -\\
&- 8\nu i\frac{\partial \omega_3}{\partial \bar{z}} - 4\alpha\omega_+ + G_+ = 0,\\
&(k + \nu)\Delta u_3 + (\gamma + \varepsilon)\Delta\omega_3 + 2\alpha i\left(\frac{\partial \bar{u}_+}{\partial \bar{z}} - \frac{\partial u_+}{\partial z}\right) + 4\alpha i\left(\frac{\partial \bar{\omega}_+}{\partial \bar{z}} - \frac{\partial \omega_+}{\partial z}\right) - 4\alpha\omega_3 + G_3 = 0,
\end{aligned}$$

where

$$u_+ = u_1 + iu_2,\quad \omega_+ = \omega_1 + i\omega_2,\quad \theta_1 = \frac{\partial \bar{u}_+}{\partial \bar{z}} + \frac{\partial u_+}{\partial z},\quad \theta_2 = \frac{\partial \bar{\omega}_+}{\partial \bar{z}} + \frac{\partial \omega_+}{\partial z},\quad \Delta = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}.$$

General representations for the components of the displacement and the microrotation vectors have the following form

$$2\mu u_+ = \frac{\lambda+3\mu}{\lambda+2\mu}\varphi(z) - \frac{\lambda+\mu}{\lambda+2\mu} z\overline{\varphi'(z)} - \frac{\mu(\beta+2\gamma)(\lambda+2\mu)(\delta+2k)-\mu(\delta+2k)^3}{\alpha(\lambda+2\mu)^2}\frac{\partial\chi}{\partial\overline{z}} -$$

$$-\frac{4\mu l_{11}}{\lambda_1} i \frac{\partial\chi_1}{\partial\overline{z}} - \frac{4\mu l_{12}}{\lambda_2} i \frac{\partial\chi_2}{\partial\overline{z}} + \mu\overline{\psi(z)},$$

$$2\mu u_3 = \left[\frac{\nu\mu - k\nu - 2k\alpha}{\alpha} l_{11} + \frac{\nu^2 - k^2 - \gamma\alpha - \varepsilon\alpha}{\alpha} l_{21}\right]\chi_1 +$$

$$+\left[\frac{\nu\mu - k\nu - 2k\alpha}{\alpha} l_{12} + \frac{\nu^2 - k^2 - \gamma\alpha - \varepsilon\alpha}{\alpha} l_{22}\right]\chi_2 +$$

$$+\frac{k(\mu-\nu-2\alpha)}{\alpha\mu} i\left(\overline{\varphi'(z)} - \varphi'(z)\right) + f(z) + \overline{f(z)},$$

$$2\alpha\omega_+ = \left[\frac{\delta\mu - k\lambda - \nu\lambda}{\mu(\lambda+2\mu)} + \frac{k(\mu-\nu-2\alpha)+2\nu\alpha}{\mu^2}\right]\overline{\varphi''(z)} - \frac{\alpha}{\mu} i\overline{f'(z)} + \left[\beta + 2\gamma - \frac{(\delta+2k)^2}{\lambda+2\mu}\right]\frac{\partial\chi}{\partial\overline{z}} +$$

$$+\left[\frac{k(\mu+\nu+2\alpha)-2\nu(\mu+\alpha)}{\mu} l_{11} + \frac{(\gamma+\varepsilon)(\mu+\alpha)+k^2-3\nu^2-2\nu k}{\mu} l_{21}\right] i\frac{\partial\chi_1}{\partial\overline{z}} +$$

$$+\left[\frac{k(\mu+\nu+2\alpha)-2\nu(\mu+\alpha)}{\mu} l_{12} + \frac{(\gamma+\varepsilon)(\mu+\alpha)+k^2-3\nu^2-2\nu k}{\alpha} l_{22}\right] i\frac{\partial\chi_2}{\partial\overline{z}},$$

$$2\alpha\omega_3 = [(\mu+\alpha)l_{11} + (k+\nu)l_{21}]\chi_1 + [(\mu+\alpha)l_{12} + (k+\nu)l_{22}]\chi_2 + \frac{\alpha}{\mu} i\left(\overline{\varphi'(z)} - \varphi'(z)\right),$$

where $\varphi(z)$, $\psi(z)$ and $f(z)$ are holomorfic functions of z, $\chi(z,\overline{z})$, $\chi_1(z,\overline{z})$ and $\chi_2(z,\overline{z})$ are the general solutions of Helmholtz's equations

$$\Delta\chi - \eta^2\chi = 0, \quad \Delta\chi_1 - \eta_1^2\chi_1 = 0, \quad \Delta\chi_2 - \eta_2^2\chi_2 = 0$$

$$\eta^2 = \frac{4\alpha(\lambda+2\mu)}{(\beta+2\gamma)(\lambda+2\mu)-(\delta+2k)^2}, \quad \eta_1^2 = \lambda_1, \quad \eta_2^2 = \lambda_2,$$

λ_1, λ_2 and $\vec{l}_1 = (l_{11}, l_{21})$, $\vec{l}_2 = (l_{12}, l_{22})$ are the eigenvalue numbers and the eigenvalue vectors of $A^{-1}B$ matrix, where

$$A = \begin{pmatrix} (\mu+\alpha)(k\mu + 2\mu\nu - k\nu - 2k\alpha) & 2\mu\nu(\nu+k) + \alpha(\nu^2 - k^2 - (\mu+\alpha)(\gamma+\varepsilon)) \\ (k+\nu)(\mu\nu - k\nu - 2\alpha k) + \mu(\mu+\alpha)(\gamma+\varepsilon) & (k+\nu)(\nu^2 + \mu\gamma + \mu\varepsilon - k^2 - \gamma\alpha - \varepsilon\alpha) \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & -4\mu\alpha^2 \\ 4\mu^2\alpha & -4\alpha\mu(\nu-k) \end{pmatrix}.$$

Complex combination of force stresses and couple stresses are represented as follows

$$\tau_{11} - \tau_{22} + i(\tau_{12} + \tau_{21}) = A_{11} z\overline{\varphi''(z)} + A_{12}\overline{\varphi'''(z)} + A_{13}\overline{\psi'(z)} + A_{14} i\overline{f''(z)} + A_{15}\frac{\partial^2\chi}{\partial\overline{z}^2} +$$

$$+ A_{16} i\frac{\partial^2\chi_1}{\partial\overline{z}^2} + A_{17} i\frac{\partial^2\chi_2}{\partial\overline{z}^2},$$

$$\tau_{11}+\tau_{22}+i(\tau_{12}-\tau_{21})=A_{21}\left(\varphi'(z)+\overline{\varphi'(z)}\right)+A_{22}\chi(z,\bar{z})+A_{23}i\chi_1(z,\bar{z})+A_{24}i\chi_2(z,\bar{z}),$$

$$\tau_{+}=A_{31}i\overline{\varphi''(z)}+A_{32}\overline{f'(z)}+A_{33}i\frac{\partial\chi}{\partial\bar{z}}+A_{34}\frac{\partial\chi_1}{\partial\bar{z}}+A_{35}\frac{\partial\chi_2}{\partial\bar{z}},$$

$$\mu_{11}-\mu_{22}+i(\mu_{12}+\mu_{21})=A_{41}z\overline{\varphi''(z)}+A_{42}\overline{\varphi'''(z)}+A_{43}\overline{\psi'(z)}+A_{44}i\overline{f''(z)}+A_{45}\frac{\partial^2\chi}{\partial\bar{z}^2}+$$

$$+A_{46}i\frac{\partial^2\chi_1}{\partial\bar{z}^2}+A_{47}i\frac{\partial^2\chi_2}{\partial\bar{z}^2},$$

$$\mu_{51}+\mu_{22}+i(\mu_{12}-\mu_{21})=A_{51}\left(\varphi'(z)+\overline{\varphi'(z)}\right)+A_{52}\chi(z,\bar{z})+A_{53}i\chi_1(z,\bar{z})+A_{54}i\chi_2(z,\bar{z}),$$

$$\mu_{+}=A_{61}i\overline{\varphi''(z)}+A_{62}\overline{f'(z)}+A_{63}i\frac{\partial\chi}{\partial\bar{z}}+A_{64}\frac{\partial\chi_1}{\partial\bar{z}}+A_{65}\frac{\partial\chi_2}{\partial\bar{z}}.$$

Let suppose that the beginning of coordinate system coincides with the centre of the circle and let $\vec{F}$ and $\vec{G}$ are constants.

Boundary conditions have the following form

$$u_r+iu_\theta=iu_+\frac{d\bar{z}}{ds}=0,\quad r=R,\qquad \omega_r+i\omega_\theta=i\omega_+\frac{d\bar{z}}{ds}=0,\quad r=R,$$
$$u_3=0,\qquad r=R,\qquad\qquad \omega_3=0,\qquad r=R. \tag{3}$$

For $u_+, u_3, \omega_+, \omega_3$ we have

$$2\mu u_+=d_{11}\varphi(z)+d_{12}z\overline{\varphi'(z)}+d_{13}\frac{\partial\chi}{\partial\bar{z}}+d_{14}i\frac{\partial\chi_1}{\partial\bar{z}}+d_{15}i\frac{\partial\chi_2}{\partial\bar{z}}+d_{16}\overline{\psi(z)}+$$

$$+\frac{\lambda+3\mu}{4(\lambda+2\mu)}F_+z\bar{z}-\frac{\lambda+\mu}{8(\lambda+2\mu)}\overline{F}_+z^2+\left(\frac{\nu-k}{2\alpha}iF_3-\frac{P_3}{2}i\right)z,$$

$$2\mu u_3=d_{21}\chi_1+d_{22}\chi_2+d_{23}i\left(\overline{\varphi'(z)}-\varphi'(z)\right)+f(z)+\overline{f(z)}+\frac{k(\mu-\nu-2\alpha)i}{4\alpha\mu}\left(z\overline{F}_+-\bar{z}F_+\right)-$$

$$-\frac{\nu\mu-k\nu-2k\alpha}{2\mu\alpha}P_3+\frac{(\nu-k)(\nu\mu-k\nu-2k\alpha)+\mu\left(\nu^2-k^2-\gamma\alpha-\varepsilon\alpha\right)+4\alpha^2\mu}{2\alpha^2\mu}F_3,$$

$$2\alpha\omega_+=d_{31}\overline{\varphi''(z)}+d_{32}i\overline{f'(z)}+d_{33}\frac{\partial\chi}{\partial\bar{z}}+d_{34}i\frac{\partial\chi_1}{\partial\bar{z}}+d_{35}i\frac{\partial\chi_2}{\partial\bar{z}}+\frac{1}{4}\left[\frac{\delta+2k}{\lambda+2\mu}+\frac{k\nu+2k\alpha-\nu\mu}{\mu^2}\right]F_+-\frac{P_+}{2},$$

$$2\alpha\omega_3=d_{41}\chi_1+d_{42}\chi_2+d_{43}i\left(\overline{\varphi'(z)}-\varphi'(z)\right)+\frac{\alpha}{4\mu}i\left(z\overline{F}_+-\bar{z}F_+\right)+\frac{2\mu\nu+\alpha\nu-\alpha k}{2\mu\alpha}F_3-\frac{\mu+\alpha}{2\mu}P_3.$$

Let as introduced functions $\varphi(z), \psi(z), f(z), \chi(z,\bar{z}), \chi_1(z,\bar{z}), \chi_2(z,\bar{z})$ by series

$$\varphi(z)=\sum_{k=0}^{\infty}a_kz^k,\qquad \psi(z)=\sum_{k=0}^{\infty}b_kz^k,\qquad f(z)=\sum_{k=0}^{\infty}c_kz^k,$$

$$\chi(z,\bar{z})=\sum_{-\infty}^{+\infty}\alpha_kI_k(\eta r)e^{ik\theta},\quad \chi_1(z,\bar{z})=\sum_{-\infty}^{+\infty}\beta_kI_k(\eta_1 r)e^{ik\theta},\quad \chi_2(z,\bar{z})=\sum_{k=-\infty}^{+\infty}\gamma_kI_k(\eta_2 r)e^{ik\theta} \tag{4}$$

where $I_k(\eta r)$, $I_k(\eta_1 r)$ and $I_k(\eta_2 r)$ are Bessel's functions. By substituting (3) into (4) we obtain the system of equations, those solutions, which are not zero, are

a_1, a_2, b_0, c_0, c_1, α_1, α_{-1}, β_0, β_1, β_{-1}, γ_0, γ_1, γ_{-1}.

Thus

$$\varphi(z) = a_1 z + a_2 z^2, \quad \psi(z) = b_0, \quad f(z) = c_0 + c_1 z,$$
$$\chi(z,\bar{z}) = \alpha_1 I_1(\eta r) e^{i\theta} + \alpha_{-1} I_1(\eta r) e^{-i\theta},$$
$$\chi_1(z,\bar{z}) = \beta_0 I_0(\eta_1 r) + \beta_1 I_1(\eta_1 r) e^{i\theta} + \beta_{-1} I_1(\eta_1 r) e^{-i\theta},$$
$$\chi_2(z,\bar{z}) = \gamma_0 I_0(\eta_2 r) + \gamma_1 I_1(\eta_2 r) e^{i\theta} + \gamma_{-1} I_1(\eta_2 r) e^{-i\theta}.$$

R E F E R E N C E S

1. Natroshvili D., Giorgashvili L. and Stratis I.G. Mathematical Problems of the Theory of Elasticity of Hemitropic Materials. Applied Mathematics Informatics and Mechanics, Tbilisi, vol. 8, p.47-103, 2003.
2. Muskhelishvili N. I. Some Basic Problems of the Mathematical Theory of Elasticity, Noordhoff, Groningen, Holland, 1953.
3. Vekua I. N. Theory of Thin and Shallow Shells with Variable Thickness, Tbilisi, ,,Metsniereba", 1965 (Russian).