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ON THE STABILITY AND CONVERGENCE OF A SYMMETRIC WEIGHTED SEMIDISCRETE SCHEME FOR DYNAMIC EQUATIONS OF A SPHERICAL SHELL

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1. Statement of the Problem

Using I. Vekua's theory (zero approximation) (see [1]), we consider the equilibrium equations of a spherical shell for the dynamic case:

$$\frac{\partial^2 u}{\partial t^2} + Au = f(x,y,t), \quad (x,y,t) \in \Omega \times]0,T[\,, \tag{1.1}$$

where

$$A = A_0 + A_1 =$$

$$= -\sigma_0 \begin{pmatrix} \frac{2(1-\sigma)}{1-2\sigma}\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \varepsilon^2 & \frac{1}{1-2\sigma}\cdot\frac{\partial^2}{\partial x \partial y} & 0 \\ \frac{1}{1-2\sigma}\cdot\frac{\partial^2}{\partial x \partial y} & \frac{\partial^2}{\partial x^2} + \frac{2(1-\sigma)}{1-2\sigma}\frac{\partial^2}{\partial y^2} - \varepsilon^2 & 0 \\ 0 & 0 & \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - 4\varepsilon^2\frac{1}{1-2\sigma} \end{pmatrix} -$$

$$-\sigma_0 \begin{pmatrix} 0 & 0 & -\varepsilon\frac{3-2\sigma}{1-2\sigma}\cdot\frac{\partial}{\partial x} \\ 0 & 0 & -\varepsilon\frac{3-2\sigma}{1-2\sigma}\cdot\frac{\partial}{\partial y} \\ \varepsilon\frac{3-2\sigma}{1-2\sigma}\cdot\frac{\partial}{\partial x} & \varepsilon\frac{3-2\sigma}{1-2\sigma}\cdot\frac{\partial}{\partial y} & 0 \end{pmatrix},$$

with the homogeneous Dirichlet boundary and Cauchy initial conditions:

$$u(x,y,t)\big|_{\partial\Omega} = 0, \quad \partial\Omega : |x| = |y| = 1, \tag{1.2}$$

$$u(x,y,0) = \varphi_0(x,y), \quad u_t'(x,y,0) = \varphi_1(x,y), \tag{1.3}$$

where $f = (f_1, f_2, f_3)^T$, $\varphi_0 = (\varphi_{01}, \varphi_{02}, \varphi_{03})^T$ and $\varphi_1 = (\varphi_{11}, \varphi_{12}, \varphi_{13})^T$ are the known continuous vector functions, $u = (u_1, u_2, u_3)^T$ is the twice continuously differentiable function we want to define, $\varepsilon = 2R^{-1}H$, H is the semithickness of the shell, R is the sphere radius, σ is Poisson's ratio, E is Young's modulus, $\sigma_0 = E/(1+\sigma)$.

To solve problem (1.1–3), we use the following semidiscrete scheme (see [2]):

$$\frac{u_{k+1} - 2u_k + u_{k-1}}{\tau^2} + A_0 \frac{u_{k+1} + \nu u_k + u_{k-1}}{2+\nu} + A_1 u_k = f(x,y,t_k), \quad k = 1,2,\ldots,n-1, \quad (1.4)$$

where $\tau = T/n$ ($n \geq 2$ is a natural number), $t_k = k\tau$, $\nu \neq -2$; by $u(x,y,t)$ we denote the exact solution $u_k(x,y)$ at the point $t = t_k$.

From (1.4) we obtain

$$\left(I + \frac{\tau^2}{2+\nu} A_0\right) u_{k+1} - \left(2I - \frac{\tau^2 \nu}{2+\nu} A_0\right) u_k + \left(I + \frac{\tau^2}{2+\nu} A_0\right) u_{k-1} =$$

$$= -\tau^2 A_1 u_k + \tau^2 f(x,y,t_k), \quad k = 1,2,\ldots,n-1. \quad (1.5)$$

It is known (see [3]) that the operator A_0 is symmetric and positive definite. Hence it follows that for $2+\nu > 0$ there exists the operator $\left(I + \frac{\tau^2}{2+\nu} A_0\right)^{-1}$ and it is bounded.

When $k = 1$, we have $u_0(x,y) = \varphi_0(x,y)$. We can find u_1 by using the Taylor formula with the initial condition (1.3) and equation (1.1) taken into account:

$$u_1 = \varphi_0 + \tau\varphi_1 + \frac{\tau^2}{2}\left(f(x,y,0) - A\varphi_0\right).$$

Thus at each time step the solution of problem (1.1 – 3) reduces to the solution of an equation of the form

$$\left(I + \frac{\tau^2}{2+\nu} A_0\right) u(x,y) = f(x,y) \quad (1.6)$$

with the homogeneous Dirichlet boundary condition

$$u(x,y)\big|_{\partial\Omega} = 0, \quad \partial\Omega : |x| = |y| = 1. \quad (1.7)$$

2. Stability Theorem

We introduce the following spaces:

$L_2(\Omega)$ is the space of square-integrable function (Hilbert space) in the domain Ω;

$H = [L_2(\Omega)]^3$ is the Hilbert space with the scalar product

$$((u,v)) = (u_1,v_1) + (u_2,v_2) + (u_3,v_3),$$

and the norm

$$\|u\| = \left(\|u_1\|_{L_2}^2 + \|u_2\|_{L_2}^2 + \|u_3\|_{L_2}^2\right)^{1/2},$$

where $u=(u_1,u_2,u_3)$ is the vector function with components from the space $L_2(\Omega)$; $(\cdot\,,\cdot)$ and $\|\cdot\|_{L_2}$ are respectively the scalar product and the norm in the Hilbert space $L_2(\Omega)$;

$C^2(\overline{\Omega})$ is the space of twice continuous differentiable functions in the closure of the domain Ω;

$[C^2(\overline{\Omega})]^3$ is the space of twice continuous differentiable vector functions in $\overline{\Omega}$.

The definition domain of the operator A_0 has the form

$$D(A_0)=\{u\in [C^2(\overline{\Omega})]^3 : u|_{\partial\Omega}=0\}.$$

We denote by $\tilde{A}_0$ the extension of the symmetric, positive definite operator A_0 to the self-conjugate operator. Since, as we know, $\tilde{A}_0$ is positive definite, there exists a square root $\tilde{A}_0^{1/2}$.

We have the following theorem.

Theorem 1. *If the vector functions* $u_0(\cdot\,,\cdot)$ *and* $u_1(\cdot\,,\cdot)$ *belong to the domain of definition of the operator* $\tilde{A}_0$ *and the vector functions* $f(\cdot\,,\cdot\,,t_i)$ *are square summable and* $\nu\in\,]-2,2[$, *then the following a priori estimates hold for scheme* (1.4):

$$\|u_{k+1}\|\le a_{k-1}\left[(c_0+\tau\varepsilon c_3)\|u_0\|+(c_1+\tau^2\varepsilon c_3)\left\|\frac{\Delta u_0}{\tau}\right\|\right]+ \\ +c_2\tau\sum_{i=1}^{k}a_{k-i}\left\|\tilde{A}_0^{-1/2}f(\cdot\,,\cdot\,,t_i)\right\|, \tag{2.1}$$

$$\left\|\frac{\Delta u_k}{\tau}\right\|\le(1+t_k a_{k-1})\left[(c_4+\tau\varepsilon c_3)\left\|\tilde{A}_0^{1/2}u_0\right\|+c_0\left\|\frac{\Delta u_0}{\tau}\right\|+ \right. \\ \left. +\tau(\sigma_1+\tau\varepsilon c_3)\left\|\tilde{A}_0^{1/2}\frac{\Delta u_0}{\tau}\right\|+c_2\tau\sum_{i=1}^{k}\|f(\cdot,\cdot,t_i)\|\right], \tag{2.2}$$

$$\left\|\tilde{A}_0^{1/2}u_{k+1}\right\|\le a_{k-i}\left[(c_0+\tau\varepsilon c_3)\left\|\tilde{A}_0^{1/2}u_0\right\|+\tau\left(\frac{1}{2}c_2+\tau\varepsilon c_3\right)\left\|\tilde{A}_0^{1/2}\frac{\Delta u_0}{\tau}\right\|+c_0\left\|\frac{\Delta u_0}{\tau}\right\|\right]+ \\ +c_2\tau\sum_{i=1}^{k}a_{k-i}\|f(\cdot\,,\cdot\,,t_i)\|, \tag{2.3}$$

where $k=1,2,\ldots,n-1$, $\Delta u_k=u_{k+1}-u_k$; c_0, c_1, c_2, c_3 *and* c_4 *are positive constants depending on* ν, $c_3=c\,c_2$, $c=\dfrac{3-2\sigma}{1-2\sigma}$, $a_k=\exp(\varepsilon c_3 t_k)$.

Proof. The following inequality is valid:

$$\|A_1u\|^2\le\varepsilon^2c^2((A_0u,u)),\quad \forall u\in D(A_0). \tag{2.4}$$

Indeed, we have

$$\| A_1 u \|^2 = \varepsilon^2 c^2 \left(\| \partial_x u_3 \|_{L_2}^2 + \| \partial_y u_3 \|_{L_2}^2 + \| \partial_x u_1 + \partial_y u_2 \|_{L_2}^2 \right), \tag{2.5}$$

$$\begin{aligned}((A_0 u, u)) = &\left(\| \partial_x u_1 \|_{L_2}^2 + \| \partial_y u_1 \|_{L_2}^2 \right) + \left(\| \partial_x u_2 \|_{L_2}^2 + \| \partial_y u_2 \|_{L_2}^2 \right) + \\ &+ \left(\| \partial_x u_3 \|_{L_2}^2 + \| \partial_y u_3 \|_{L_2}^2 \right) + b \| \partial_x u_1 + \partial_y u_2 \|_{L_2}^2 + \\ &+ \varepsilon^2 \left(\| u_1 \|_{L_2}^2 + \| u_2 \|_{L_2}^2 + 4b \| u_3 \|_{L_2}^2 \right),\end{aligned} \tag{2.6}$$

where $b = \dfrac{1}{1-2\sigma}$.

Equalities (2.5) and (2.6) clearly imply inequality (2.4).

From (2.4) we have

$$\| \tilde{A} u \| \le \varepsilon c \| \tilde{A}_0^{1/2} \|, \quad \forall u \in D\left(\tilde{A}_0^{1/2}\right) \subset D\left(\tilde{A}_1\right) \subset H . \tag{2.7}$$

If the conditions of Theorem 1 are fulfilled, then the following estimates are valid (see [4], Ch. III):

$$\| u_{k+1} \| \le c_0 \| u_0 \| + c_1 \left\| \frac{\Delta u_0}{\tau} \right\| + c_2 \tau \sum_{i=1}^{k} \left\| \tilde{A}_0^{-1/2} f(\cdot, \cdot, t_i) \right\| + c_2 \tau \sum_{i=1}^{k} \left\| \tilde{A}_0^{-1/2} A_1 u_i \right\|, \tag{2.8}$$

$$\left\| \frac{\Delta u_k}{\tau} \right\| \le c_2 \| \tilde{A}_0^{1/2} u_0 \| + c_0 \left\| \frac{\Delta u_0}{\tau} \right\| + c_2 \tau \sum_{i=1}^{k} \| f(\cdot, \cdot, t_i) \| + c_2 \tau \sum_{i=1}^{k} \| A_1 u_i \|, \tag{2.9}$$

$$\begin{aligned}\| \tilde{A}_0^{1/2} u_{k+1} \| \le c_0 &\left(\| \tilde{A}_0^{1/2} u_0 \| + \left\| \frac{\Delta u_0}{\tau} \right\| \right) + \frac{1}{2} c_2 \tau \left\| \tilde{A}_0^{1/2} \frac{\Delta u_0}{\tau} \right\| \\ &+ c_2 \tau \sum_{i=1}^{k} \| f(\cdot, \cdot, t_i) \| + c_2 \tau \sum_{i=1}^{k} \| A_1 u_i \|.\end{aligned} \tag{2.10}$$

Taking into account the relation $\left(\tilde{A}_1 \tilde{A}_0^{-1/2}\right)^* \supset \tilde{A}_0^{-1/2} \tilde{A}_1^* \supset \tilde{A}_0^{-1/2} \tilde{A}_1$, from inequalities (2.8) and (2.7) we obtain

$$\| u_{k+1} \| \le c_0 \| u_0 \| + c_1 \left\| \frac{\Delta u_0}{\tau} \right\| + c_2 \tau \sum_{i=1}^{k} \left\| \tilde{A}_0^{-1/2} f(\cdot, \cdot, t_i) \right\| + (\varepsilon c c_2) \tau \sum_{i=1}^{k} \| u_i \|. \tag{2.11}$$

Let us introduce the following notations:

$$\varepsilon_i = \| u_i \|, \quad i = 1, 2, \ldots, k+1,$$

$$\delta_i = c_2 \tau \left\| \tilde{A}_0^{-1/2} f(\cdot, \cdot, t_i) \right\|, \quad i = 1, 2, \ldots, k,$$

$$\delta_0 = c_0 \| u_0 \| + c_1 \left\| \frac{\Delta u_0}{\tau} \right\|, \quad c_\tau = (\varepsilon c c_0) \tau.$$

Now inequality (2.11) can be rewritten as

$$\varepsilon_{k+1} \le c_\tau \sum_{i=1}^{k} \varepsilon_i + \sum_{i=0}^{k} \delta_i \,,$$

from which, by induction, we obtain

$$\varepsilon_{k+1} \le c_\tau (1+c_\tau)^{k-1} \varepsilon_1 + (1+c_\tau)^{k-1} \delta_0 + \sum_{i=1}^{k} (1+c_\tau)^{k-1} \delta_i \,, \tag{2.12}$$

which, as is known, is a discrete analogue of Gronwall's lemma.

If we take into account that

$$(1+c_\tau)^k = (1+\varepsilon\, c\, c_3\, \tau)^k \le e^{\varepsilon c_3 t_k} \,,$$

then (2.12) implies estimate (2.1).

Estimates (2.2) and (2.3) are proved in a similar manner.

3. Convergence Theorems

Let us introduce the following spaces.

We define the Hermite norm $\| u \|_2 = \| \tilde{A}_0 u \|$ in $D(\tilde{A}_0)$ and obtain the Hilbert space, which we denote by W^2. In a similar manner we define the norm $\| u \|_1 = \| \tilde{A}_0^{1/2} u \|$ in $D(\tilde{A}_0^{1/2})$ and obtain the Hilbert space denoted by W^1.

We denote by $C([0,T];H)$ the set of continuous on the interval $[0,T]$ vector functions $f(\cdot\,,\cdot\,,t)$ having values from *H*.

We denote by $C^m([0,T];H)$ $(m \ge 1)$ the set of continuously differentiable on $[0,T]$ vector functions up to order *m* inclusive from $C([0,T];H)$.

In a similar manner we consider $C([0,T];W^\lambda)$ and $C^m([0,T];W^\lambda)$, $\lambda = 1,2$.

We introduce the notation

$$z_k(x,y) = u(x,y,t_k) - u_k(x,y), \quad k = 1,2,\ldots,n\,.$$

The following theorem holds true.

Theorem 2. *Let* $u_0 = \varphi_0$, $u_1 = \varphi_0 + \tau\varphi_1$, $\varphi_0, \varphi_1 \in W^2$, $f(\cdot\,,\cdot\,,t) \in C([0,T];H)$, $\nu \in \,]-2,2[$. *Then*

a) *if problem* (1.1–3) *has solutions* $u(\cdot\,,\cdot,t) \in C^2([0,T];H) \cap C([0,T];W^2)$, *then*

$$\max_{1\le k\le n-1} \left(\| z_{k+1} \| + \left\| \frac{\Delta z_k}{\tau} \right\| \right) \to 0 \quad \text{as} \quad \tau \to 0\,;$$

b) *if* $f(\cdot\,,\cdot\,,t) \in C^1([0,T];H)$ *or* $f(\cdot\,,\cdot\,,t) \in C([0,T];W^2)$, *then*

$$\max_{1\le k\le n-1}\left(\| z_{k+1} \| + \left\| \frac{\Delta z_k}{\tau} \right\| + \left\| \tilde{A}_0^{1/2} z_{k+1} \right\| \right) \to 0 \quad \text{as} \quad \tau \to 0\,;$$

c) *if* $f(\cdot\,,\cdot\,,t) \in C^1([0,T];H)$ *and* $u(\cdot\,,\cdot\,,t) \in C^3([0,T];H) \cap C([0,T];W^2)$, *then*

$$\max_{1\le k\le n-1}\left(\| z_{k+1} \| + \left\| \frac{\Delta z_k}{\tau} \right\| + \left\| \tilde{A}_0^{1/2} z_{k+1} \right\| \right) \le c_5 \tau, \quad c_5 = \text{const} > 0\,.$$

Theorem 3. *Let* $u_0 = \varphi_0$, $\varphi_0 \in W^2$, $u_1 = \varphi_0 + \tau\varphi_1 + \dfrac{\tau^2}{2}\left(f(\cdot\,,\cdot\,,0) - \left(\tilde{A}_0\varphi_0 + \tilde{A}_1\varphi_0 \right) \right)$, φ_1, $\tilde{A}_0\varphi_0$, $f(\cdot,\cdot,0) \in W^2$, $f(\cdot,\cdot,t) \in C^2([0,T];H)$, *and* $\nu \in \,]-2,2[$. *Then*

a) *if problem* (1.1–3) *has solutions* $u(\cdot\,,\cdot\,,t) \in C^4([0,T];H) \cap C([0,T];W^2)$, *then*

$$\max_{1\le k\le n-1} \| z \| \le c_6 \tau^2, \quad c_6 = \text{const} > 0\,;$$

b) *if* $u(\cdot\,,\cdot\,,t) \in C^4([0,T];H) \cap C^2([0,T];W^1) \cap C([0,T];W^2)$, *then*

$$\max_{1\le k\le n-1}\left(\left\| \frac{\Delta z_k}{\tau} \right\| + \left\| \tilde{A}_0^{1/2} z_{k+1} \right\| \right) \le c_7 \tau^2, \quad c_7 = \text{const} > 0\,.$$

The theorems given in this subsection are resulted from Theorem 1.

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