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ON THE NUMERICAL INTEGRATION OF ONE SYSTEM OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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In some problems of mathematical physics the following system of differential equations appears [1]:

$$\frac{\partial U}{\partial t} = f\left(x,t,U,V,\frac{\partial U}{\partial x},\frac{\partial^2 U}{\partial x^2}\right), \quad (x,t)\in \Omega\times(0,T], \tag{1}$$

$$\frac{\partial V}{\partial t} = g\left(x,t,U,V\right), \quad (x,t)\in \overline{\Omega}\times(0,T], \tag{2}$$

where $f = f(x,t,u,v,p,q)$ and $g = g(x,t,u,v)$ are given functions of their arguments, $\frac{\partial f}{\partial q} \ge 0$, $U = U(x,t)$ and $V = V(x,t)$ are unknown functions, $T = const > 0$, $\Omega = (0,1)$.

The solution of the system (1),(2) is considered under the following initial-boundary conditions:

$$U(x,0) = U_0(x), \quad x\in\overline{\Omega}; \quad U(0,t) = \phi_1(t), \; U(1,t) = \phi_2(t), \; t\in(0,T], \tag{3}$$

$$V(x,0) = V_0(x), \quad x\in\overline{\Omega}. \tag{4}$$

The problem similar to (1)-(4) was studied in the works [2]-[6] and in a number of other works as well.

In the work [2] for the case

$$f\left(x,t,u,v,\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial x^2}\right) = a(x,t)\frac{\partial^2 u}{\partial x^2} + h(x,t,u,v)$$

the problem (1)-(4) and its multidimensional analogue are investigated.

In the case

$$f\left(x,t,u,v,\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial x^2}\right) = a(x,t,u)\frac{\partial^2 u}{\partial x^2} + b(x,t,u)\frac{\partial u}{\partial x} + c(x,t,u,v)$$

the problem (1)-(4) was studied by the finite difference method in the work [3].

This problem (1)-(4) and its multidimensional analogue for the case

$$f\left(x,t,u,v,\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial x^2}\right) = a\left(x,t,u,\frac{\partial u}{\partial x}\right)\frac{\partial^2 u}{\partial x^2} + b\left(x,t,u,\frac{\partial u}{\partial x}\right)$$

in the papers [4],[5] are investigated.

The comparison theorems for the case of different coefficients and different initial-boundary data if

$$f\left(x,t,u,v,\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial x^2}\right)=a(u)\frac{\partial^2 u}{\partial x^2}+b(u,v), \quad g(x,t,u,v)=h(u,v)$$

in the work [6] are proved.

The initial-boundary value problems for nonlinear parabolic equations are studied in the works [7],[8],[9] and in a number of other works as well.

In this work the convergence and existence of the solution of the difference scheme for problem (1) – (4) are proved.

Enter a grid on area $\Omega\times[0,T]$ on variables x and t respectively: $\omega_h=\{x_k=kh,\, h>0,\, k=0,1,...,M;\, hM=1\}$, $\omega_\tau=\{t_j=j\tau,\, \tau>0,\, j=0,1,...,N;\, \tau N=T\}$.

Put to a problem (1)- (4) in conformity implicit discrete analogue:

$$\frac{u_i^{j+1}-u_i^j}{\tau}=f\left(x_i,t^{j+1},u_i^{j+1},v_i^{j+1},\frac{u_{i+1}^{j+1}-u_{i-1}^{j+1}}{2h},\frac{u_{i+1}^{j+1}-2u_i^{j+1}+u_{i-1}^{j+1}}{h^2}\right),\ i=1,2,...,M-1, \tag{5}$$

$j=0,1,...,N-1$,

$$\frac{v_i^{j+1}-v_i^j}{\tau}=g\left(x_i,t^{j+1},u_i^{j+1},v_i^{j+1}\right),\ i=0,1,...,M, \tag{6}$$

$$u_i^0=U_{0,i},\ i=0,1,...,M;\ u_0^j=\phi_1^j,\ u_M^j=\phi_2^j,\quad j=1,2,...,N, \tag{7}$$

$$v_i^0=V_{0,i},\ i=0,1,...,M. \tag{8}$$

Theorem 1: *Let* $\left|\frac{\partial f}{\partial u}\right|\le L,\ \left|\frac{\partial f}{\partial v}\right|\le L,\left|\frac{\partial f}{\partial p}\right|\le L,\left|\frac{\partial g}{\partial u}\right|\le L,\ \left|\frac{\partial g}{\partial v}\right|\le L,\ \frac{\partial f}{\partial q}\ge\delta>0$, *there exists solution* $\{U,V\}$ *of the problem (1) - (4) such, that* $\left|\frac{\partial^2 U}{\partial t^2}\right|\le M$, $\left|\frac{\partial^4 U}{\partial x^4}\right|\le M$, $\left|\frac{\partial^2 V}{\partial t^2}\right|\le M$. *Let also, exists an unique solution of a problem (5), (8) for small values* τ *and* h. *Then the following estimation takes place:*

$$\|u-U\|_C+\|v-V\|_C\le C\left(\tau+h^2\right).$$

Let's construct for solving equations (5), (6) iterative circuit:

$$\overset{k+1}{u_i}=(1-\theta)\overset{k}{u_i}+\theta\left[\tau f\left(x,t,\overset{k}{u_i},\overset{k}{v_i},\frac{\overset{k}{u}_{i+1}-\overset{k}{u}_{i-1}}{2h},\frac{\overset{k}{u}_{i+1}-2\overset{k}{u_i}+\overset{k}{u}_{i-1}}{h^2}\right)+u_i^j\right], \tag{9}$$

$$\overset{k+1}{v_i}=(1-\theta)\overset{k}{v_i}+\theta\left[\tau g\left(x,t,\overset{k}{u_i},\overset{k}{v_i}\right)+v_i^j\right]. \tag{10}$$

where $0<\theta<1$, $\overset{0}{u_i}=u_i,\overset{0}{v_i}=v_i$.

Theorem 2: *Let* $\left|\frac{\partial f}{\partial u}\right| \le L,\ \left|\frac{\partial f}{\partial v}\right| \le L,\left|\frac{\partial f}{\partial p}\right| \le L,\left|\frac{\partial g}{\partial u}\right| \le L,\ \left|\frac{\partial g}{\partial v}\right| \le L,\ 0<\delta \le \frac{\partial f}{\partial q} \le L$. *Then if* $\tau < \frac{1}{2L}$, $h < \frac{2\delta}{L}$ *and* $0<\theta \le \left(1+\frac{2\tau L}{h^2}+\tau L\right)^{-1}$, *the solution of (9), (10) converges to the solution of equations (5), (6) and the following estimation is true:*

$$\left\|\overset{k+1}{u}-\overset{k}{u}\right\|_C+\left\|\overset{k+1}{v}-\overset{k}{v}\right\|_C=\left(1-\theta\left(1-2\tau L\right)\right)\left(\left\|\overset{k}{u}-\overset{k-1}{u}\right\|_C+\left\|\overset{k}{v}-\overset{k-1}{v}\right\|_C\right).$$

Theorem 3: *Let* $f=f\left(x,t,u,v,p,q\right)$ *and* $g=g\left(x,t,u,v\right)$ *are not decreasing functions with respect of arguments* v *and* u *respectively, and also satisfy the following conditions:*

$$\left|\frac{\partial f}{\partial u}\right| \le L,\ \left|\frac{\partial f}{\partial v}\right| \le L,\ \left|\frac{\partial f}{\partial p}\right| \le L,\ \left|\frac{\partial g}{\partial u}\right| \le L,\ \left|\frac{\partial g}{\partial v}\right| \le L,\ \frac{\partial f}{\partial q} \ge \delta > 0.$$

Let also, $\{\bar{u},\bar{v}\}$ *and* $\{\bar{\bar{u}},\bar{\bar{v}}\}$ *are two pairs of the net functions determined on a grid* $\omega_h \times \omega_\tau$, *satisfying the following inequalities:*

$$\frac{\bar{\bar{u}}_i^{j+1}-\bar{\bar{u}}_i^{j}}{\tau}-f\left(x_i,t^{j+1},\bar{\bar{u}}_i^{j+1},\bar{\bar{v}}_i^{j+1},\frac{\bar{\bar{u}}_{i+1}^{j+1}-\bar{\bar{u}}_{i-1}^{j+1}}{2h},\frac{\bar{\bar{u}}_{i+1}^{j+1}-2\bar{\bar{u}}_{i}^{j+1}+\bar{\bar{u}}_{i-1}^{j+1}}{h^2}\right)\ge$$

$$i=1,2,\dots,M-1,\quad j=0,1,\dots,N-1,$$

$$\ge\frac{\bar{u}_i^{j+1}-\bar{u}_i^{j}}{\tau}-f\left(x_i,t^{j+1},\bar{u}_i^{j+1},\bar{v}_i^{j+1},\frac{\bar{u}_{i+1}^{j+1}-\bar{u}_{i-1}^{j+1}}{2h},\frac{\bar{u}_{i+1}^{j+1}-2\bar{u}_{i}^{j+1}+\bar{u}_{i-1}^{j+1}}{h^2}\right),$$

$$\frac{\bar{\bar{v}}_i^{j+1}-\bar{\bar{v}}_i^{j}}{\tau}-g\left(x_i,t^{j+1},\bar{\bar{u}}_i^{j+1},\bar{\bar{v}}_i^{j+1}\right)\ge\frac{\bar{v}_i^{j+1}-\bar{v}_i^{j}}{\tau}-g\left(x_i,t^{j+1},\bar{u}_i^{j+1},\bar{v}_i^{j+1}\right),$$

$$i=0,1,\dots,M,\quad j=0,1,\dots,N-1,$$

and for

$\bar{\bar{u}}_i^0 \ge \bar{u}_i^0,\ \bar{\bar{v}}_i^0 \ge \bar{v}_i^0,\ i=0,1,\dots,M$,

$\bar{\bar{u}}_0^j \ge \bar{u}_0^j,\ \bar{\bar{u}}_M^j \ge \bar{u}_M^j,\ j=1,2,\dots,N$.

Then for $\tau<\frac{1}{L}$ *and* $h<\frac{2\delta}{L}$ *we have:* $\bar{\bar{u}}_i^j \ge \bar{u}_i^j, \bar{\bar{v}}_i^j \ge \bar{v}_i^j,\ i=0,1,\dots,M,\ j=0,1,\dots,N$.

Theorem 4*:* *Let* $f_k=f_k\left(x,t,u,v,p,q\right)$ *and* $g_k=g_k\left(x,t,u,v\right)$, $k=1,2$ *are two pairs the functions satisfying inequalities:*

$f_2\left(x,t,u,v,p,q\right) \ge f_1\left(x,t,u,v,p,q\right)$

$g_2\left(x,t,u,v\right) \ge g_1\left(x,t,u,v\right)$

and one of pairs $\left(f_1,g_1\right)$, $\left(f_2,g_2\right)$ *satisfy the following conditions:*

$$\left|\frac{\partial f_k}{\partial u}\right| \le L,\ \left|\frac{\partial f_k}{\partial v}\right| \le L,\ \left|\frac{\partial f_k}{\partial p}\right| \le L,\ \left|\frac{\partial g_k}{\partial u}\right| \le L,\ \left|\frac{\partial g_k}{\partial v}\right| \le L,\ \frac{\partial f_k}{\partial q} \ge \delta > 0.$$

Let also, net functions $\{\overline{u},\overline{v}\}$ *also* $\{\overline{\overline{u}},\overline{\overline{v}}\}$ *are solutions of the following difference equations:*

$$\frac{\overline{u}_i^{j+1}-\overline{u}_i^{j}}{\tau}=f_1\left(x_i,t^{j+1},\overline{u}_i^{j+1},\overline{v}_i^{j+1},\frac{\overline{u}_{i+1}^{j+1}-\overline{u}_{i-1}^{j+1}}{2h},\frac{\overline{u}_{i+1}^{j+1}-2\overline{u}_i^{j+1}+\overline{u}_{i-1}^{j+1}}{h^2}\right),\ i=1,2,\ldots,M-1,$$

$$j=0,1,\ldots,N-1$$

,

$$\frac{\overline{v}_i^{j+1}-\overline{v}_i^{j}}{\tau}=g_1\left(x_i,t^{j+1},\overline{u}_i^{j+1},\overline{v}_i^{j+1}\right),\ i=0,1,\ldots,M\ ;$$

$$\frac{\overline{\overline{u}}_i^{j+1}-\overline{\overline{u}}_i^{j}}{\tau}=f_2\left(x_i,t^{j+1},\overline{\overline{u}}_i^{j+1},\overline{\overline{v}}_i^{j+1},\frac{\overline{\overline{u}}_{i+1}^{j+1}-\overline{\overline{u}}_{i-1}^{j+1}}{2h},\frac{\overline{\overline{u}}_{i+1}^{j+1}-2\overline{\overline{u}}_i^{j+1}+\overline{\overline{u}}_{i-1}^{j+1}}{h^2}\right),\ i=1,2,\ldots,M-1,$$

$$j=0,1,\ldots,N-1$$

,

$$\frac{\overline{\overline{v}}_i^{j+1}-\overline{\overline{v}}_i^{j}}{\tau}=g_2\left(x_i,t^{j+1},\overline{\overline{u}}_i^{j+1},\overline{\overline{v}}_i^{j+1}\right),\ i=0,1,\ldots,M\ ,$$

and

$\overline{\overline{u}}_i^0\geq\overline{u}_i^0,\ \overline{\overline{v}}_i^0\geq\overline{v}_i^0,\ i=0,1,\ldots,M$,

$\overline{\overline{u}}_0^j\geq\overline{u}_0^j,\ \overline{\overline{u}}_M^j\geq\overline{u}_M^j,\ j=1,2,\ldots,N$.

Then for $\tau<\frac{1}{L}$ *and* $h<\frac{2\delta}{L}$ *we have:* $\overline{\overline{u}}_i^j\geq\overline{u}_i^j,\ \overline{\overline{v}}_i^j\geq\overline{v}_i^j,\ i=0,1,\ldots,M,\ \ j=0,1,\ldots,N$.

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