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THREE-LAYER DIFFERENCE SCHEMES FOR MULTIDIMENSIONAL PARABOLIC EQUATION OF THE SECOND ORDER

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Abstract

The paper deals with the algorithm for construction of absolutely stable and conditionally approximable non-explicit three-layer schemes for the parabolic type equation with the variable coefficients. This algorithm gives us a possibility for reduction of discrete problem to the solution of a system of linear equations defined by a three-point equation. In the special case, when the dimension p =1, it is obtained absolutely stable scheme with the exactness O (τ^2 + h ^{2}).

I. For simplicity initially we consider the heat conduction equation

$$\frac{\partial u}{\partial t} = Lu + f\,,\;\; Lu = \sum_{i=1}^{p} L_i u\,, L_i u = \frac{\partial^2 u}{\partial x_i^{\,2}}\,,\quad x\in G,\quad t\in (0,T\,], \tag{1.1}$$

$$u\big|_{\Gamma} = 0,\quad u(x,0) = u_0(x),\quad (x = x_1,...,x_p)$$

Let G=Gop, be the p - dimensional cube $(0 \le x_i \le 1,\quad i = \overline{1,p}\quad)$ $\overline{\omega}_h = \{(i\,h_1,...,i\,h_p)\in G\}$ be the cube type net with the step h=1/Ni. for each x_i , i= $\overline{1,p}$, ω_τ be the net on the interval $0 \le t \le T$ with the step $\tau = \dfrac{T}{N_0}$.

The basic idea for the obtaining of difference schemes is the representation of the given differential equation in the following identity

$$\beta\,\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial t^2} = \left(\frac{\partial^2 u}{\partial t^2} + \beta\,\frac{\partial^2 u}{\partial x_k^{\,2}}\right)_{(\delta_k)} + \beta\sum_{\substack{i=1\\ i\ne k}}^{p}\frac{\partial^2 u}{\partial x_i^{\,2}} + \beta\; f\,, \tag{1.2}$$

$$k = \overline{1,p},\quad \beta = const \ne 0$$

The parabolic sections with respect to the coordinate planes passing through the elementary nucleus are denoted by δ_k (this is the basic node).

Further the index indicates δ_k mentioned Summand by means of directional derivatives.

Justifying the problem (1.1) we use (1.2) by the representation and consider the following functional

$$I(u)=\int_{t_0}^{t_n}\{\int_0^1[\,\beta u\frac{\partial u}{\partial t}+\left(\frac{\partial u}{\partial t}\right)^2+\frac{1}{2n_1}\sum_{i=1}^{2n_1}[\,A_1^{(i)}\left(\frac{\partial u}{\partial l_i}\right)^2+B_1^{(i)}\left(\frac{\partial u}{\partial l_{i+1}}\right)^2+$$

$$+C_1^{(i)}\frac{\partial u}{\partial l_i}\frac{\partial u}{\partial l_{i+1}}]_{(\delta_k)}+\beta\sum_{\substack{i=1\\ i\neq k}}^{2n_1}\left(\frac{\partial u}{\partial x_i}\right)^2-2\beta\, f\, u\,]\,dx\,\}dt$$

If the function u(x,t) is the solution of the problem (1.1), then according to the well known Hamilton principle

$$I\,(u)\to \min \qquad (1.3)$$

i.e. for the function u the functional I(u) admits his minimal value by changing of time t in an arbitrary interval. In particular, if we divide the interval of time changing tn – t0 (t0=0) into the small intervals tj – tj-1 , j=$\overline{1,n}$. Then it is possible to rewrite the condition (1.3) in the following form:

$$I(u)=\sum_{j=1}^{n}I_j(u)\to min$$

where

$$I_j(u)=\int_{t_{j-1}}^{t_j}\{\int_0^1[\,\beta u\frac{\partial u}{\partial t}+\left(\frac{\partial u}{\partial t}\right)^2+\frac{1}{2n_1}\sum_{i=1}^{2n_1}[\,A_1^{(i)}\left(\frac{\partial u}{\partial l_i}\right)^2+B_1^{(i)}\left(\frac{\partial u}{\partial l_{i+1}}\right)^2+$$

$$+C_1^{(i)}\frac{\partial u}{\partial l_i}\frac{\partial u}{\partial l_{i+1}}]_{(\delta_k)}+\beta\sum_{i=1}^{2n_1}\left(\frac{\partial u}{\partial x_i}\right)^2-2\beta\, f\, u\,]\,dx\,\}dt$$

$$A_1^{(i)}=\frac{cos^2\alpha_{i+1}+\beta\, sin^2\alpha_{i+1}}{sin^2(\alpha_{i+1}-\alpha_i)},\qquad B_1^{(i)}=\frac{cos^2\alpha_i+\beta\, sin^2\alpha_i}{sin^2(\alpha_{i+1}-\alpha_i)},$$

$$C_1^{(i)}=\frac{-2(\,cos\,\alpha_i\,cos\,\alpha_{i+1}+\beta\,sin\,\alpha_{i+1}\,sin\,\alpha_i)}{sin^2(\alpha_{i+1}-\alpha_i)},$$

The directions l_i are defined by means of the angles α_i in the following way

$$\vec{l}=(l_1,...,l_{2n_1})\Rightarrow(\alpha_1,...,\alpha_{2n_1}),\qquad \alpha_{n_{1+i}}=\pi+\alpha_i,\quad l_i=-l_{n_{1+i}},\quad i=\overline{1,n}$$

$$(\int_0^1=\int_0^1...\int_0^1)$$

If we repeat the inference, which is reduced in [1], we obtain the following difference problem

$$u_{\overset{0}{t}}+\tau^2R_k u_{t\bar{t}}=L_h u+f,\quad u\big|_{\Gamma_n}=0,\quad u(x,0)=u_0(x)$$

$$L_h u=\sum_{i=1}^{p}u_{x_i\bar{x}_i},\qquad R_k u=-\sigma u_{x_k\bar{x}_k},\qquad k=\overline{1,p} \qquad (1.4)$$

(we remain the same notations for difference functions).

Analogously we can write the difference scheme for the equation

$$\frac{\partial u}{\partial t}=Lu+f,$$

where $L=\sum_{i,j=1}^{p}a_{ij}\frac{\partial^2}{\partial x_i\partial x_j}-\sum_{i=1}^{p}a_i\frac{\partial}{\partial x_i}-a$ is the elliptic operator.

$$\left(C_1 \sum_{i=1}^{p} \xi_i^{\ 2} \quad \le \sum_{i,j=1}^{p} a_{i,j} \xi_i \xi_j \quad \le c_2 \sum_{i=1}^{p} \xi_i^{\ 2} \right)$$

Moreover we will mean that

$$\sqrt{\sum_{i=1}^{p} a_i^{\ 2}}, \qquad |a| < c_2, \qquad c_1, c_2 = const > 0$$

$$u_t^0 + \tau^2 R_k u_{t\bar{t}} = L_h u + f, \tag{1.5}$$

where

$$L_h u = \sum_{i=1}^{p} a_{ii} u_{x_i \bar{x}_i} + \sum_{\substack{i,j=1 \\ i \ne j}}^{p} a_{ij} \left[\frac{1}{4} \left(u_{x_i} + u_{\bar{x}_i} \right)_{x_j} + \frac{1}{4} \quad \left(u_i + u_{\bar{x}_i} \right)_{\bar{x}_{jj}} \right] \quad +$$

$$+ \sum_{i=1}^{p} a_i u_{x_i}^0 \quad + au, \qquad R_k = \sigma a_{kk} L_k, \qquad L_k = -\Delta_{kk}, \quad -\Delta_{kk} u = -u_{x_k \bar{x}_k}, \quad k = \overline{1, p}$$

(1.5) represent one dimensional p amount and independent from one another equivalent difference schemes. For the realization of these schemes it is applied the sweep method with respect to the axes OXk

II. The investigation of the schemes (1.4), (1.5) is based on the general principle of regularization.

The sufficient stability condition for canonical three-layer difference schemes is given in the following simple form (see [2])

$$R > \frac{1}{4} L \ \ or \ \ R \ge \frac{1+\varepsilon}{4} L \tag{2.1}$$

If we apply the inequality

$$8p \| u \|^2 \le \left(-\sum_{i=1}^{p} \Delta_{ii} u_1, u \right) \le \frac{4p}{h^2} \quad \| u \|^2, \tag{2.2}$$

for on arbitrary p (see [2], ch. IV , § 4, article 2), according to (1.1) we obtain that

$$\delta > \frac{1}{4} + \frac{p-1}{8h^2} \tag{2.3}$$

We can study the question of the stability of the difference scheme (1.5), if instead of the inequality (1.2) we apply with the inequality (1.2), considered in [2], and obtain that

$$\delta > \frac{1}{8} \frac{c_2}{c_1} \frac{p}{h^2}, \tag{2.4}$$

Thus, if we choose the parameter σ from the conditions (2.3), (2.4) the difference schemes (1.4), (1.5) represent absolutely stable schemes.

The error of approximation (t=0) is defined in the following way.

$$\Psi = L_h u - u_t^0 - \tau^2 R_k u_{t\bar{t}} = Lu - \dot{u} - \sigma \tau^2 \frac{\partial^2}{\partial x_k^2} \left(\frac{\partial^2 u}{\partial t^2} \right) + 0(\tau^2 + |h|^2) =$$

$$= 0(\tau^2 + |h|^2 - \frac{1}{8} \frac{c_2}{c_1} \frac{p\tau^2}{h^2})$$

It is obvious from this that if $\left|\frac{\partial^2}{\partial x_k^2}\left(\frac{\partial^2 u}{\partial t^2}\right)\right|$ < M, M=const >0 is independent from τ, the schemes (1.4), (1.5) have the conditional approximation if τ= O(h^2). But if p=1, the difference scheme (1.4) is absolutely stable with the exactness O(τ^2 + h ^{2}).

With the help of numerical methods we verity that the same exactness reminds also if τ=O(h) , $(h \le c\tau, \quad \frac{1}{2} \le c \le 1)$

For the comparison, in the case p=l and R=E, the conditional approximate explicit scheme of Diufort – Franchel is well known, which is obtained from the Richardson scheme u_t^0 =Δ11U by addition to the left hand $\frac{\tau^2}{h^2} u_{t\bar{t}}$. In our case it is obtained the absolutely stable scheme with the exactness O(τ^2 +h ^{2}).
where R= - Δ11

Remarks:

1. Both the factored difference schemes and the schemes (1.5) can be applied as an iterative algorithm for the difference analogy, which corresponds to the elliptic equation. For exampl see [3].
2. As we have note, (1.5) is the p amount independent from each other equivalent schemes. If we compose arithmetic mean

$$u = \frac{1}{p}\sum_{i=1}^{p} u_i \tag{2.5}$$

or the moment t=2τ
where u1,...,up are the solutions of the boundary problems corresponding to the difference schemes (1.5) and so on until Noτ, it is proved that (2.5) presents the solution for the absolutely stable with the exactness O(τ^2 +h^2) sheme

$$u_t^0 + \tau^2 \sum_{i=1}^{p} R_k\, u_{t\bar{t}} \quad = L_h u \quad + f$$

The numerical analysis conducting on the whole series of tests showed us that the process is stable, but the flow of numbers are increased argumented p times.

This algorithm is suggested by D. Gordeziani and my great respect to him.

R E F E R E N C E

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