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SIMULATION OF SOME PROBLEMS OF HYDRO DYNAMICS WITH THE USE OF NURBS SURFACES

Meladze H., Chanturia A., Karalashvili M.

I.Vekua Institute of Applied Mathematics
Tbilisi State University

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Introduction

In nowadays the construction and exploitation of hydroelectric power stations with regulating reservoirs is of great importance for every country. Natural and handmade reservoirs represent potential danger to catastrophic events (the consequence of different kinds of landslide events causes formation of gravitational waves; dam may be collapsed by natural (seismic) or handmade (sabotage, act of terrorism) events), and so, for mountainous countries, where mentioned reservoirs are located mainly in mountainous and semi-mountainous regions or for countries located in seismic regions, possibility of development of landslide events or of dam collapse is significantly increased. To evacuate population and valuables located downstream in time in such emergency cases, clarity of prediction of water masses propagation characteristics - motion velocity, front shape, expansion surface etc. is vivid. Presented work is dedicated to research of mentioned concrete problems.

Mathematical model of wave formation and propagation can be constructed on the background of common hydrodynamic equations with boundary conditions considering presence of water free surface. This paper refers to the finite difference methods of solution of hydrodynamics system of equations - a first order system of quasi-linear hyperbolic equations with non-continuous solutions, describing motion of continuous, compressible environment.

At the current stage of computer simulation and computer graphics development, a great deal of interest represents *Scientific Visualization* - or in other words, visualization of numeric data with the use of computer graphics. We also discuss simulation of dam collapse on the base of approximation of numerical results received by solution of two dimensional mathematical model of breaking wave formation and propagation when dam collapses with the use of Non Uniform Rational B-Splines, namely NURBS surfaces.

1. NURBS Surfaces – Definition and their usage in Mathematical Models

Non Uniform B-Spine commonly referred to as NURBS are CAD/CAM industry standard tools for representation and design of geometry. The enormous success behind NURBS is largely due to the fact that [14]

- NURBS provide a unified mathematical basis for representing both analytic shapes, such as conic sections and quadric surfaces, as well as free-form entities, such as car bodies and ship hulls etc.;
- NURBS algorithms are fast and numerically stable;
- NURBS curves and surfaces are invariant under common geometric transformations, such as translation, rotation, parallel and perspective projections;
- NURBS are generalizations of non-rational B-Splines and rational and non-rational Bézier curves and surfaces.

However, one of the drawbacks NURBS have is the need for extra storage to define traditional shapes (e.g. spheres). This results from parameters in addition to the control points, but finally allows the desired flexibility for defining parametric shapes.

NURBS-shapes are uniquely defined by set of control points and weights, associated with each control point.

A NURBS-surface $\vec{S}(u,\upsilon)$, of degree p in the u parametric direction and degree q in the υ parametric direction is bivariate vector-valued piecewise rational function of the form [14]:

$$\vec{S}(u,\upsilon)=\frac{\sum_{i=0}^{N}\sum_{j=0}^{M}B_{i,p}(u)B_{j,q}(\upsilon)w_{i,j}\vec{Q}_{i,j}}{\sum_{i=0}^{N}\sum_{j=0}^{M}B_{i,p}(u)B_{j,q}(\upsilon)w_{i,j}} \qquad 0\le u,\upsilon\le 1, \tag{1.1}$$

where, $\{\vec{Q}_{i,j}\}\in R^3, i=0,...,N; j=0,...,M$ form the bidirectional set of control points, the $\{w_{i,j}\}$ are the weights associated with control points, and $\{B_{i,p}(u)\},\{B_{j,q}(\upsilon)\}$ are the non-rational B-Spline basis functions defined on the knot vectors $U=\{0,...,0,u_{p+1},...,u_{r-p-1,}1,...,1\}$, $r=N+p+1$, $V=\{0,...,0,\upsilon_{q+1},...,\upsilon_{l-q-1,}1,...,1\}$, $l=M+q+1$ .with the following recursive formulas

$$B_{i,0}(u)=\begin{cases}1, & u\in D_1=[u_i,u_{i+1})\\ 0, & u\notin D_1\end{cases} \tag{1.2}$$

$$B_{i,p}(u)=\frac{u-u_i}{u_{i+p}-u_i}B_{i,p-1}(u)+\frac{u_{i+p+1}-u}{u_{i+p+1}-u_{i+1}}B_{i+1,p-1}(u). \tag{1.3}$$

$$B_{j,0}(\upsilon)=\begin{cases}1, & \upsilon\in D_2=[\upsilon_j,\upsilon_{j+1})\\ 0, & \upsilon\notin D_2\end{cases} \tag{1.4}$$

$$B_{j,q}(\upsilon)=\frac{\upsilon-\upsilon_j}{\upsilon_{j+q}-\upsilon_j}B_{j,q-1}(\upsilon)+\frac{\upsilon_{j+q+1}-\upsilon}{\upsilon_{j+q+1}-\upsilon_{j+1}}B_{j+1,q-1}(\upsilon). \tag{1.5}$$

In order to simulate mathematical model of hydro dynamics for different time moments a NURBS surfaces are constructed over difference solutions of mathematical model.

1.1. Why NURBS ?

- As it can be seen from (1.2)-(1.5) formulas, instead of being fixed by the number of control points, the set of blending functions $\{B_{i,p}(u)\},\{B_{j,q}(\upsilon)\}$ may be set to any degree $d_p - 1, d_q - 1$ desired appropriately, where $2 \le d_p < N+1, 2 \le d_q < M+1$. So, order of the surface in any direction (u or υ) can be taken more flexibly.

- Also, as mentioned above, the $R_{i,j}(u,\upsilon)$ are step functions, equal to zero everywhere except the rectangle given by $\lfloor u_i, u_{i+p+1}) \times \lfloor \upsilon_j, \upsilon_{j+q+1})$,what means, that local control over the control points can be achieved – we can use both control point and weight modification to locally change the shape of NURBS surface.
- NURBS give shape preserving approximation.

2. Two-Dimensional Mathematical Model of Breaking Wave Formation and Propagation when Dam Collapses

In this chapter two-dimensional mathematical problem of adjacent dam collapse will be solved and received numerical results approximated with the use of NURBS surfaces.

Two dams are called adjacent, if they are located in canyons outgoing to the same plane. Two dimensional mathematical problem is stated as follows: at the initial moment of time, dams located at the upper right and lower left corners of computational area are instantly collapsed, thus, two powerful flows moving towards each other are formed. Computational domain of a given problem is square $D = \{0 \le x \le a, 0 \le y \le a\}$.

The basis of a mathematical model of breaking wave formation and propagation when dam collapses is the assumption, that vertical component of water particles acceleration slightly affects the pressure, or in other words pressure distribution in fluid obeys the hydrostatic low (shallow water theory). Considering the water as an ideal fluid and the water stream potential, water flow can be described by Saint-Venant equations. In addition it is considered, that water is of constant depth as in upper so in lower pool of dam.

At the initial moment of time motion of breaking waves formed when dam collapses, is characterized with sharp water level changing on short section and with curved profile of free surface. Consequent propagation of breaking wave causes comparably small changes of levels in lower pool.

If for any time moment as for unknown variables are considered depth of riverbed, or water level, and for given cutting plane (averaged) velocity distribution, water motion into channel or in open riverbed can be described by Saint-Venant system of differential equations.

Let us write corresponding two dimensional mathematical model in divergence form for unknown impulse's $J_1(x,y,t)$ and $J_2(x,y,t)$ components and $H(x,y,t)$ flow depth. In vector form it has following form:

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} F_1(Q) + \frac{\partial}{\partial y} F_2(Q) = W(Q), \tag{2.1}$$

where, $J_1 = Hu$, $J_2 = H\upsilon$, $Q = (J_1,\ J_2,\ H)^T$,

$$F_1(Q) = (J_1 u + gH^2/2,\ J_1\upsilon,\ J_1)^T,$$

$$F_2(Q) = (J_2 u,\ J_2\upsilon + gH^2/2,\ J_2)^T,$$

$$W(Q) = (f_1(x,y,t,H,u,\upsilon),\ f_2(x,y,t,H,u,\upsilon),\ f_3(x,y,t,H,u,\upsilon))^T,$$

$$H(x,y,0)=\begin{cases} a_1 & (x,y)\in D_1, \\ a_2 & (x,y)\in D_2, \\ a_3 & (x,y)\in D_3. \end{cases} \tag{2.2}$$

$$u(x,y,0)=\upsilon(x,y,0)=0, \quad (x,y)\in D=D_1\cup D_2\cup D_3, \tag{2.3}$$

x and y are Euler variables, t–is time variable, g-acceleration of gravity, $u(x,y,t)$ and $\upsilon(x,y,t)$-projections of velocity vector on Ox and Oy axes accordingly. $a_1, a_2, a_3 = const$, $f_1(x,y,t,H,u,\upsilon)$, $f_2(x,y,t,H,u,\upsilon)$, $f_3(x,y,t,H,u,\upsilon)$ - functions, describing sources of mass and impulse.

3. Numerical Experiments

In order to numerically solve (2.1)-(2.3) two dimensional mathematical model, two layer difference scheme with non-linear regularizing functional $R_i(Q)$, $i=1,2$, which do not changes order of scheme and has a meaning of "artificial viscosity", will be used 20:

$$Q_{ht}+\left[F_{h1}(Q_h,\sigma)\right]_{\overset{\circ}{x}}+\left[F_{h2}(Q_h,\sigma)\right]_{\overset{\circ}{y}}=R_{h1}(Q_h)+R_{h2}(Q_h)+W_h(Q_h), \tag{3.1}$$

$$\begin{aligned} &H_h(ih,jh,0)=H_0(ih,jh), \\ &u_h(ih,jh,0)=u_0(ih,jh),\ \upsilon_h(ih,jh,0)=\upsilon_0(ih,jh). \end{aligned} \tag{3.2}$$

where, grid functions

$$Q_h=\left(J_{h1},\ J_{h2,}\ H_h\right)^T,\ W_h(Q_h)=\left(f_{h1}, f_{h2}, f_{h3}\right),$$

$$F_{h1}(Q_h)=\left(J_{h1}u_h+0.5gH_h^{\ 2},\ J_{h1}\upsilon_h,\ J_{h1}\right)^T,$$

$$F_{h2}(Q)=\left(J_{h2}u_h,\ J_{h2}\upsilon_h+0.5gH_h^{\ 2},\ J_{h2}\right)^T,$$

where, τ and h are steps in time and space accordingly,

$$r_{h1}(Q_h)=0.25H_h\left((\lambda_{11}+\lambda_{12})u_h^{(\sigma)}+0.5c_h(\lambda_{11}-\lambda_{12}),(\lambda_{11}+\lambda_{12})\upsilon_h^{(\sigma)},\ (\lambda_{11}+\lambda_{12})\right)^T,$$

$$r_{h1}(Q_h)=0.25H_h\left((\lambda_{21}+\lambda_{22})u_h^{(\sigma)},(\lambda_{21}+\lambda_{22})\upsilon_h^{(\sigma)}+0.5c_h(\lambda_{21}-\lambda_{22}),\ (\lambda_{21}+\lambda_{22})\right)^T,$$

where , $\lambda_{11}=|u_h+c_h|$, $\lambda_{12}=|u_h-c_h|$, $\lambda_{21}=|\upsilon_h+c_h|$, $\lambda_{22}=|\upsilon_h-c_h|$, $c_h=\sqrt{gH_h}$ - is a difference analogue of velocity 's perturbation propagation, $\sigma=\hat{H}_h^{1/2}\Big/\left(\hat{H}_h^{1/2}+H_h^{1/2}\right)$; $\varphi_k(\tau,h), k=1,2$ are sufficiently smooth functions of its variables and $\varphi_k(\tau,h)\to 0$, $k=1,2$ when $\tau\to 0, h\to 0$, $\varphi_k(\tau,h)=O(h^2)$; grid functions $H_h=H(ih,ji,k\tau)$, $u_h=u(ih,ji,k\tau)$, $\upsilon_h=\upsilon(ih,ji,k\tau)$, $f_{lh}=f_l(ih,ji,k\tau)$, $l=1,2,3$ are defined on $D=\varpi_{h^2}\times\varpi_\tau$ area and approximate H,u,υ,f_l , $l=1,2,3$ functions accordingly. $H_0(ih,jh)$, $u_0(ih,jh)$, $\upsilon_0(ih,jh)$ are difference analogues of depth and velocity components distribution functions correspondingly.

Problem Statement*:* Two dams of same height are located in opposite canyons of same valley. At the initial moment of time dams are instantly collapsed and two powerful water flows moving towards each other are formed. To describe dynamic of water flows (2.1)-(2.3) problem is solved with (3.1)-(3.2) difference scheme. As a computational domain is taken $D=\{0\le x\le 10, 0\le y\le 10\}$ square; dams are modeled as planes, projections of which are represented by lines in upper and lower corners of square. Initial conditions are of following form:

$$H(x,y,0)=\begin{cases} 10, & (x,y)\in D_1 \\ 10, & (x,y)\in D_2 \\ 1, & (x,y)\in D\setminus\{D_1\cup D_2\}, \end{cases}$$

$$u(x,y,0)=\upsilon(x,y,0)=0,\ \ (x,y)\in D,$$

where, $D_1=\{(x,y):\ x\in[0,2.5]\}$, $D_2=\{(x,y):\ x\in[7.5,10], y\in[17.5-x,10]\}$.

In boundary conditions presence of solid walls are taken into consideration by zeroth component of velocity vector on corresponding boundary points of considered domain. Computation in both spatial directions are made with step $h=0.1$ in conditions, that $f_1(x,y,t,H,u,\upsilon)=f_2(x,y,t,H,u,\upsilon)=f_3(x,y,t,H,u,\upsilon)=0$. Relation between spatial and time steps is chosen from CFL condition:

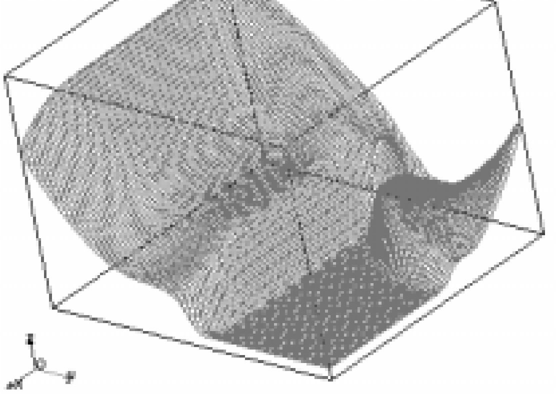
$$\tau\le Cr\cdot h\Big/\max_i\{(|u_i|+c),\ (|\upsilon_i|+c)\}$$

where, Cr is a Courant number.

Results of numerical calculations for flow's H depth and u velocity vector as NURBS surfaces are shown on figures 4-6 for time moments:

Fig. 1 $t=0.3\,sec.$

a)



b)

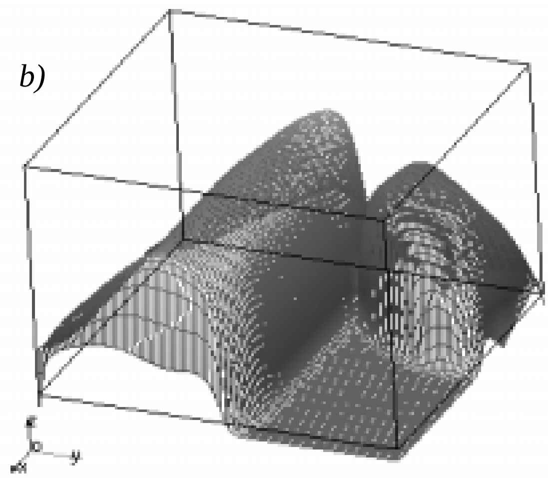
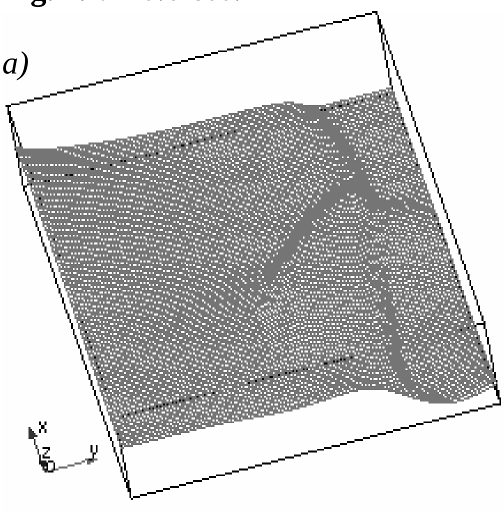


Fig. 2. $t=0.6\ sec.$

a)



b)

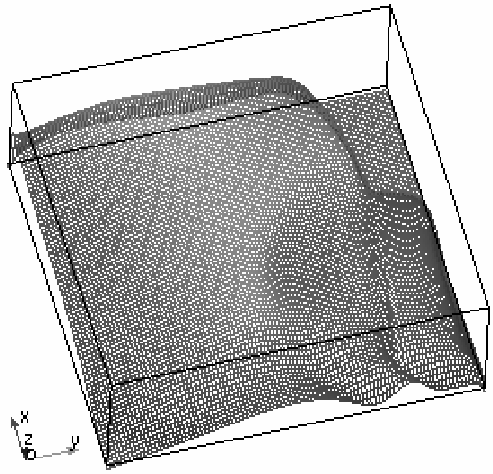
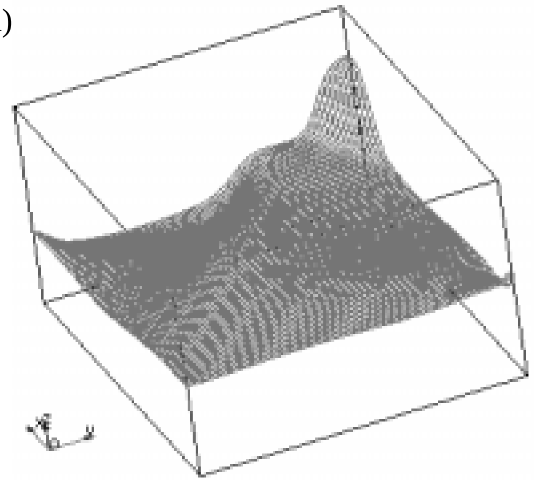
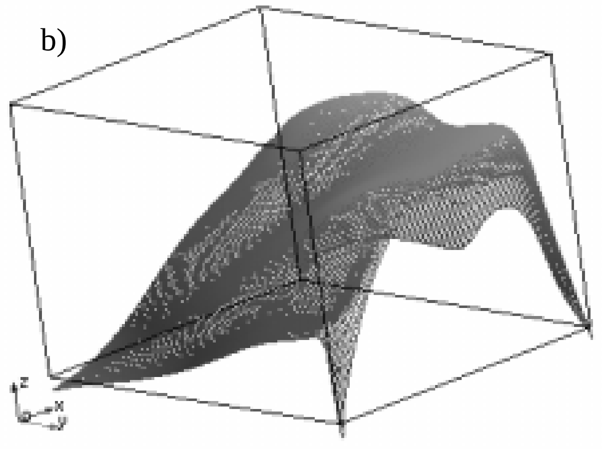


Fig. 3. $t = 1.25$ *sec*

a)



b)



After collision time moment $t = 0.3\,sec.$ (fig. 1) the bigger volume water flow ingests oppositely moving second flow. Breaking wave propagates towards free corner of computational area (Fig. 2.), (Fig.3) and after reflection keeps moving towards opposite corner

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