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ASYMPTOTIC BEHAVIOR AND NUMERICAL SOLUTION OF THE SYSTEM OF NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS

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1. Asymptotic behavior.

Let us consider the following nonlinear integro-differential system:

$$\frac{\partial U}{\partial t}-\frac{\partial}{\partial x}\left(a(S)\frac{\partial U}{\partial x}\right)=0,\ \frac{\partial V}{\partial t}-\frac{\partial}{\partial x}\left(a(S)\frac{\partial V}{\partial x}\right)=0, \qquad (1)$$

where

$$S(x,t)=1+\int_0^t\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial V}{\partial x}\right)^2\right]d\tau$$

and a is given function, which satisfy the following conditions $a(S)\geq a_0=const>0$, $a'(S)\geq 0, \quad a''(S)\leq 0$.

In the cylinder $(0,1)\times(0,\infty)$ for the system (1) let us consider the following initial-boundary value problem:

$$U(0,t)=U(1,t)=V(0,t)=V(1,t)=0,\ t\geq 0, \qquad (2)$$

$$U(x,0)=U_0(x), V(x,0)=V_0(x),\ \ x\in[0,1]. \qquad (3)$$

By the system of equations (1) the penetration of the electromagnetic field in the substance is described [1].

Many scientifically works are devoted to the investigation of the proposed problem (1)-(3) (see, e. g. [1]-[12]). In our work we have received such a priori estimations of the given problem, from which the solution stabilization derives when $t\to\infty$.

The following statement takes place.

Theorem1. *If* $a(S)\geq a_0=const>0$, $a'(S)\geq 0, \quad a''(S)\leq 0$, $U_0,V_0\in \overset{o}{W}{}^1_2(0,1)$, *then for the solution of the problem (1)-(3) the following estimate is true*

$$\|U\|^2+\left\|\frac{\partial U}{\partial x}\right\|^2+\left\|\frac{\partial U}{\partial t}\right\|^2+\|V\|^2+\left\|\frac{\partial V}{\partial x}\right\|^2+\left\|\frac{\partial V}{\partial t}\right\|^2\leq Ce^{-\omega t},$$

where $\omega<a_0$.

The proof of the Theorem 1 is based on some Lemmas.

Lemma 1. *If* $a(S) \geq a_0 = const > 0$ *then for the solution of the problem (1)-(3) the following estimate is true*

$$e^{2a_0 t}\left(\|U\|^2 + \|V\|^2\right) \leq C.$$

Here and below C denote positive constants dependent only on U_0, V_0 and consequently independent from t.

Lemma 1 gives the stabilization of the solution of the problem (1)-(3) when $t \to \infty$. Besides, in the sence of $L_2(0,1)$ space norm the rate of stabilization is exponential [4].

Lemma 2. *For the solution (1)-(3) the following estimate is hold*

$$e^{\omega t}\left[\sigma_1\left(\|U\|^2 + \|V\|^2\right) + \sigma_2\left(\left\|\frac{\partial U}{\partial x}\right\|^2 + \left\|\frac{\partial V}{\partial x}\right\|^2\right) + \sigma_3\left(\left\|\frac{\partial U}{\partial t}\right\|^2 + \left\|\frac{\partial V}{\partial t}\right\|^2\right)\right] +$$

$$+\sigma_4\left(\int_0^t e^{\omega\tau}\left\|\frac{\partial U}{\partial \tau}\right\|^2 d\tau + \int_0^t e^{\omega\tau}\left\|\frac{\partial V}{\partial \tau}\right\|^2 d\tau\right) +$$

$$+\sigma_5\left(\int_0^t e^{\omega\tau}\int_0^1 a'(S)\left(\frac{\partial U}{\partial x}\right)^4 dx d\tau + \int_0^t e^{\omega\tau}\int_0^1 a'(S)\left(\frac{\partial V}{\partial x}\right)^4 dx d\tau\right) \leq C,$$

where $\omega < 2a_0$, $\sigma_i \in R$, $i = 1,2,...5.$

Lemma 3. *It exists numbers* $\omega < a_0$, γ, ε', ε, μ, $\eta \in R^+$ *such, that the following inequalities are true:*

$$\sigma_1 = \left(\frac{\omega a_0 \gamma}{2a_0 - \omega} - \frac{\gamma(\varepsilon + \omega)}{2} - \mu\varepsilon' + \eta\right) > 0,$$

$$\sigma_2 = \gamma a_0 > 0,$$

$$\sigma_3 = \left(1 - \frac{\gamma}{2\varepsilon} - \frac{\mu}{4\varepsilon'}\right) > 0,$$

$$\sigma_4 = (2a_0 - \omega - 2\mu) > 0,$$

$$\sigma_5 = \mu - \frac{\omega}{2} > 0.$$

From the Lemmas 1, 2 and 3 the validity of the Theorem 1 is received.

2. Difference scheme and numerical solution.

Here is received the convergence of the difference scheme for the given problem. Note that the discrete analogues for the problem of such type are also studied in the [5],[6].

Let us introduce the notations $E = S^{\frac{\alpha+1}{2}}$, $\alpha \neq -1$ and reduce the problem (1)-(3) to the following form:

$$\frac{\partial U}{\partial t}-\frac{\partial}{\partial x}\left(E^{\delta}\frac{\partial U}{\partial x}\right)=0,\qquad \frac{\partial V}{\partial t}-\frac{\partial}{\partial x}\left(E^{\delta}\frac{\partial V}{\partial x}\right)=0,$$

$$\frac{\partial E}{\partial t}=\frac{1}{2-\delta}E^{\delta-1}\left[\left(\frac{\partial U}{\partial x}\right)^{2}+\left(\frac{\partial V}{\partial x}\right)^{2}\right], \tag{4}$$

$$U(0,t)=V(0,t)=U(1,t)=V(1,t)=0,$$

$$U(x,0)=U_0(x),\quad V(x,0)=V_0(x),\ E(x,0)=[S_0(x)]^{\frac{1}{2-\delta}},$$

where $\delta=\frac{2\alpha}{\alpha+1}$.

Let us construct the grids on $[0,1]\times[0,T]$ and follow to the standard notations and for the problem (4) consider the following difference scheme:

$$u_t^j=\left(e^{\delta}u_{\bar{x}}\right)_x,\ v_t^j=\left(e^{\delta}v_{\bar{x}}\right)_x,$$

$$e_t^j=\frac{1}{2-\delta}e^{\delta-1}\left(u_{\bar{x}}^2+v_{\bar{x}}^2\right),$$

$$u_0^j=v_0^j=u_N^j=v_N^j=0,\quad j=0,1,...M, \tag{5}$$

$$u_i^0=U_0(x_i),\ v_i^0=V_0(x_i),\ \ i=0,1,...N,$$

$$e_i^0=[S_0(x_{i+1/2})]^{\frac{1}{2-\delta}},\ i=0,1,...N-1.$$

It is easy to establish that the rate of approximation of the difference scheme (5) on the smooth solutions of the problem (4) is $O(\tau+h^2)$.

The following statement is true.

Theorem 2. *If* $|\delta|\le 1$, *then the rate of approximation of the difference scheme (5) is* $O(\tau+h^2)$ *and the solution of this scheme converges to the solution of the problem (4) with rate* $O(\tau+h^2)$ *in the sence of the discrete norm* L_2.

Note that to solve the difference scheme is used the modified Newton's iterative process and the various numerical experiments are done. These experiments agree with theoretical research.

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