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ON NUMERICAL RESOLUTION AND UNIQUENESS OF SOLUTION OF INITIAL-BOUNDARY VALUE PROBLEM FOR THE GENERALIZED CHARNEY-OBUKHOV EQUATION

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Abstract

In the present work, first order accuracy implicit difference schemes for the numerical solution of the nonlinear generalized Charney-Obukhov equation with scalar nonlinearity is constructed. On the basis of numerical calculations accomplished by means of these schemes, the dynamics of a two-dimensional nonlinear solitary Rossby vortex structure is studied. In addition, for the considered equation the theorem of uniqueness of the solution in case of periodic boundary conditions is proved.

Key words and phrases: Solitary Rossby Wave, Charney-Obukhov Equation, Hasegava-Mima Equation, Implicit Difference scheme.
AMS subject classification: A10,85A30,76N10,76Q05.

Introduction

Numerical investigation of Charney-Obukhov (CO) nonlinear equation describing Rossby wave in geophysical streams and its analogous Hasegava-Mima (HM) nonlinear equation describing propagation of drift wave in laboratory plasma immersed in magnetic field intensively began in the eighties of the last century (see e.g. [1-7]) and owing to its urgency is being continued up to now (see the special issue of the journal Chaos, Vol. 4, No. 2, 1994 devoted to the coherent nonlinear structures in planetary atmospheres and oceans and numerous literature cited there). It should be noted that from the mathematical point of view there is no difference between these equations except for CO equation is written for a stream function but HM equation (obtained independently 30 years later) for perturbed plasma potential. Questions of stability of Rossby waves were investigated in [8-10]. But in the works known to us difference schemes of numerical calculation used for the above-mentioned equations are explicit ones, among them is Lax-Wendroff scheme (see [11], §12.6) very often used in mentioned papers. It is known that the stability factor of explicit schemes for nonlinear equations is low. For this reason it is efficient to use implicit schemes, the stability factor of which is significantly higher.

In the present work the first order accuracy implicit difference schemes for numerical solution of the generalized Charney-Obukhov (correspondingly of the generalized Hasegava-Mima) equation with scalar nonlinearity are constructed. On the basis of numerical calculations accomplished by means of these schemes dynamics of two-dimensional nonlinear solitary Rossby vortical structure is studied.

1. Statement of the problem

Nonlinear dimensionless generalized Charney-Obukhov equation containing scalar nonlinearity of the Korteweg - de Vries (KdV) type, in the frame of reference moving with velocity v along the axis OX can be written in the following form [1-5]:

$$\frac{\partial(\Delta\psi-\gamma\psi)}{\partial t}+\beta\frac{\partial\psi}{\partial x}-v\frac{\partial(\Delta\psi-\gamma\psi)}{\partial x}+J(\psi,\Delta\psi)-\alpha\psi\frac{\partial\psi}{\partial x}=0, \tag{2.1}$$

where β and γ are positive constants defined through physical characteristics of the medium; In case of drift waves the coefficient of scalar KdV nonlinearity $\alpha>0$, while in case of Rossby waves $\alpha<0$. ψ is the streamfunction, $J(\psi,\Delta\psi)$ is the Jacobian

$$J(\psi,\Delta\psi)=\frac{\partial\psi}{\partial x}\frac{\partial\Delta\psi}{\partial y}-\frac{\partial\psi}{\partial y}\frac{\partial\Delta\psi}{\partial x},$$

describing a contribution of the so called vectorial nonlinearity.
It is well-known that equation (2.1) in the absence of KdV nonlinearity describes the propagation of solitary dipole vortical structure consisting of cyclone and anticyclone. But the presence of KdV scalar nonlinearity should lead to formation of monopole vortical structure. So the final dynamics depends on the competition of these two nonlinearities.

If we introduce the generalized vorticity

$$W=\Delta\psi-\gamma\psi+\beta y,$$

the equation (2.1) deduces to the following system with respect to ψ and W:

$$\frac{\partial W}{\partial t}+\frac{\partial\psi}{\partial x}\frac{\partial W}{\partial y}-\frac{\partial\psi}{\partial y}\frac{\partial W}{\partial x}-v\frac{\partial W}{\partial x}-\alpha\psi\frac{\partial\psi}{\partial x}=0, \tag{2.2}$$

$$\Delta\psi-\gamma\psi=W-\beta y. \tag{2.3}$$

Our aim is to solve system (2.2), (2.3) numerically in the cylindrical domain $Q_T=\Omega\times\left]0,T\right[$, where Ω is the rectangle, $\Omega=\left]-a_1,a_1\right[\times\left]-a_2,a_2\right[$ (space variables x,y vary in the domain Ω and the variable t - in the interval $\left]0,T\right[$). As an initial condition at the moment $t=0$ we take well known solitary solution $\psi(x,y,0)=\psi_0(x,y)$ (see [12]):

$$\psi_0(x,y)=y\frac{r_0}{r}\begin{cases}\dfrac{J_1(\lambda r)}{\lambda^2 J_1(\lambda r_0)}-\dfrac{r}{r_0}\left(1+\dfrac{1}{\lambda^2}\right), & r<r_0,\\[2ex] -\dfrac{K_1(r)}{K_1(r_0)}, & r>r_0,\end{cases} \tag{2.4}$$

where $r=\sqrt{x^2+y^2}$, r_0 and λ are parameters. In addition the magnitude λr_0 lies in the interval $\gamma_0<\lambda r_0<\gamma_1$ where γ_0 is the first zero of the function $(\lambda r)^{-1}J_1(\lambda r)$ and γ_1 is its minimum [see 13]. Then the value r_0 may be defined from the following dispersive equation:

$$\frac{J_2(\lambda r_0)}{\lambda r_0 J_1(\lambda r_0)}=-\frac{K_2(r_0)}{r_0 K_1(r_0)}. \tag{2.5}$$

In the mentioned expressions $J_n(z)$ is the Bessel function of the first kind and $K_n(z)$ are the modified Bessel functions of the second kind. As to the boundary conditions, they will be given after the passage to the system of difference equations (see Sec.3)

2. The theorem of uniqueness for system (2.2)-(2.3)

Due to physical character of the problem (we mean e.g. the propagation of a Rossby solitary wave) we consider it quite natural to put for system (2.2)-(2.3) periodic boundary conditions of the following type (without loss of generality we assume that $]0,1[\times]0,1[$):

$$W(0,y,t)=W(1,y,t), \quad W(x,0,t)=W(x,1,t), \tag{3.1}$$

$$\psi(0,y,t)=\psi(1,y,t), \quad \psi(x,0,t)=\psi(x,1,t), \tag{3.2}$$

$$\left.\frac{\partial\psi}{\partial x}\right|_{x=0}=\left.\frac{\partial\psi}{\partial x}\right|_{x=1}, \quad \left.\frac{\partial\psi}{\partial y}\right|_{y=0}=\left.\frac{\partial\psi}{\partial y}\right|_{y=1}. \tag{3.3}$$

It is obvious that an initial condition should be added to the boundary conditions (3.1)-(3.3):

$$W(x,y,0)=W_0(x,y), \tag{3.4}$$

where $W_0(x,y)$ is a sufficiently smooth function.
The following theorem is valid:

Theorem: *If the solution to problem (2.2),(2.3),(3.1)-(3.4)* $(W;\psi)\in C^{1,1}(\overline{Q}_T)\times C^{3,0}(\overline{Q}_T)$, *then it is unique.*

Here $C^{l,m}(\overline{Q}_T)$ (l and m are natural numbers) is a set of functions continuous in $\overline{Q}_T$, which have all derivatives including order l with respect to spatial variables, all derivatives including order m with respect to time variable, and these derivatives are continuous in $C^{l,0}(\overline{Q}_T)$ (l is a natural number) is a set of functions continuous in $\overline{Q}_T$, which have all derivatives including order l with respect to spatial variables, and these derivatives are continuous in $\overline{Q}_T$.

Proof: Assume that problem (2.2),(2.3),(3.1)-(3.4) has two solutions (W,ψ) and $(\tilde{W},\tilde{\psi})$ Then the pair (u,ζ), where $u=W-\tilde{W}$ and $\zeta=\psi-\tilde{\psi}$ satisfies the following system:

$$\frac{\partial u}{\partial t}+J(\psi,u)+J(\zeta,\tilde{W})-v\frac{\partial u}{\partial x}-\alpha\zeta\frac{\partial\psi}{\partial x}-\alpha v\frac{\partial\zeta}{\partial x}=0, \tag{3.5}$$

$$\Delta\zeta-\gamma\zeta=u. \tag{3.6}$$

Multiplying both sides of equation (3.5) on u and integrating it on the domain Ω, we obtain:

$$\begin{aligned}&\frac{1}{2}\int_\Omega\frac{\partial u^2}{\partial t}dxdy+\int_\Omega J(\psi,u)udxdy+\int_\Omega J(\zeta,\tilde{W})udxdy-\frac{1}{2}v\int_\Omega\frac{\partial u^2}{\partial x}dxdy\\&-\alpha\int_\Omega\frac{\partial\psi}{\partial x}\zeta udxdy-\alpha\int_\Omega\psi\frac{\partial\zeta}{\partial x}udxdy=0.\end{aligned} \tag{3.7}$$

$$\int_\Omega\frac{\partial u^2}{\partial x}dxdy=0. \tag{3.8}$$

The following inequality holds:

$$\int_\Omega J(\psi,u)dxdy=0. \tag{3.9}$$

From (3.7), taking into account (3.8) and (3.9) equalities, we obtain:

$$\frac{1}{2}\int_\Omega\frac{\partial u^2}{\partial t}dxdy+\int_\Omega J(\zeta,\tilde{W})udxdy-\alpha\int_\Omega\frac{\partial\psi}{\partial x}\zeta udxdy-\alpha\int_\Omega\psi\frac{\partial\zeta}{\partial x}udxdy=0.$$

Obviously from here follows the inequality:

$$\frac{1}{2}\int_{\Omega}\frac{\partial u^2}{\partial t}dxdy \le \int_{\Omega}\left|J\left(\zeta,\tilde{W}\right)u\right|dxdy - |\alpha|\left(\int_{\Omega}\left|\frac{\partial \psi}{\partial x}\zeta u\right|dxdy - \int_{\Omega}\left|\psi\frac{\partial \zeta}{\partial x}u\right|dxdy\right). \tag{3.10}$$

As well-known, if we multiply both sides of equation (3.6) on ζ and integrate it on domain Ω, then we use the formula of partial integration, and take into account that function ζ satisfies periodic boundary conditions, we obtain:

$$\int_{\Omega}\left[\left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2\right]dxdy + \gamma\int_{\Omega}\zeta^2 dxdy = -\int_{\Omega}u\zeta dxdy \tag{3.11}$$

According to Schwarz inequality and ε-inequality we have:

$$\int_{\Omega}|u\zeta|dxdy \le \left(\int_{\Omega}u^2 dxdy\right)^{\frac{1}{2}}\left(\int_{\Omega}\zeta^2 dxdy\right)^{\frac{1}{2}} \le \frac{1}{2}\left(\frac{1}{\varepsilon^2}\int_{\Omega}u^2 dxdy + \varepsilon^2\int_{\Omega}\zeta^2 dxdy\right). \tag{3.12}$$

Let us choose ε so that the condition $\gamma_0 = \gamma - \frac{1}{2}\varepsilon^2 > 0$ be satisfied, then from (3.11) in view of (3.12) we obtain:

$$\int_{\Omega}\left[\left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2\right]dxdy + \gamma_0\int_{\Omega}\zeta^2 dxdy \le c_0\int_{\Omega}u^2 dxdy, \tag{3.13}$$

where $c_0 = \dfrac{1}{2\varepsilon^2}$.

Let us estimate the right-hand side of inequality (3.10), we obtain:

$$\int_{\Omega}\left|J\left(\zeta,\tilde{W}\right)u\right|dxdy \le c_1\int_{\Omega}\left(\left|\frac{\partial \zeta}{\partial x}\right| + \left|\frac{\partial \zeta}{\partial y}\right|\right)|u|dxdy, \tag{3.14}$$

where

$$c_1 = \max\left(\max_{(x,y,t)}\left(\left|\frac{\partial \tilde{W}}{\partial x}\right|\right), \max_{(x,y,t)}\left(\left|\frac{\partial \tilde{W}}{\partial y}\right|\right)\right), \quad (x,y,t)\in \overline{Q}_T.$$

From (3.14), according to Schwarz inequality, it follows:

$$\left(\int_{\Omega}\left|J\left(\zeta,\tilde{W}\right)u\right|dxdy\right) \le 2c_1^2\left(\int_{\Omega}\left(\frac{\partial \zeta}{\partial x}\right)^2 dxdy + \int_{\Omega}\left(\frac{\partial \zeta}{\partial y}\right)^2 dxdy\right)\int_{\Omega}u^2 dxdy, \tag{3.15}$$

From (3.15), in view of (3.13), it follows:

$$\left(\int_{\Omega}\left|J\left(\zeta,\tilde{W}\right)u\right|dxdy\right) \le c_2\int_{\Omega}u^2 dxdy, \quad c_2 = const > 0. \tag{3.16}$$

Let us estimate the expression placed in the brackets in the right hand-side of inequality (3.10). By the Schwarz inequality we have:

$$\int_\Omega \left| \frac{\partial \psi}{\partial x} \zeta u \right| dxdy + \int_\Omega \left| \tilde{\psi} \frac{\partial \zeta}{\partial x} u \right| dxdy \le c_2 \left(\int_\Omega \zeta^2 dxdy \right)^{\frac{1}{2}} \left(\int_\Omega u^2 dxdy \right)^{\frac{1}{2}}$$
$$+ c_3 \left(\int_\Omega \left(\frac{\partial \zeta}{\partial x} \right)^2 dxdy \right)^{\frac{1}{2}} \left(\int_\Omega u^2 dxdy \right)^{\frac{1}{2}}, \tag{3.17}$$

where

$$c_2 = \max_{(x,y,t)} \left| \frac{\partial \psi}{\partial x} \right|, \quad c_3 = \max_{(x,y,t)} |\tilde{\psi}|, \quad (x,y,t) \in \overline{Q}_T.$$

From (3.17), with account of inequalities (3.13), we obtain:

$$\int_\Omega \left| \frac{\partial \psi}{\partial x} \zeta u \right| dxdy + \int_\Omega \left| \psi \frac{\partial \zeta}{\partial x} u \right| dxdy \le c_4 \int_\Omega u^2 dxdy, \quad c_4 = const > 0. \tag{3.18}$$

From (3.10), (3.16) and (3.18), it follows:

$$\frac{dE(t)}{dt} \le cE(t), \quad c = const > 0, \tag{3.19}$$

where

$$E(t) = \int_\Omega u^2(x,y,t) dxdy.$$

As known, from (3.19) follows:

$$E(t) \le e^{ct} E(0).$$

Since according to the condition $E(0) = 0$, then

$$E(t) = \int_\Omega u^2(x,y,t) dxdy \equiv 0, \quad t \in [0,T].$$

From here it follows that $u(x,y,t) \equiv 0$ or what the same $W(x,y,t) = \tilde{W}(x,y,t)$. In view of this identity from (3.13) follows $\zeta(x,y,t) \equiv 0$. Thus uniqueness of the solution is proved.

3. The first order accuracy implicit scheme with respect to time step

Let us introduce a time step $\tau = T/m \quad (m > 1)$ and approximate equation (2.2) at the point (x,y,t_k), where $t_k = k\tau \quad (k = 1,2,...)$, by the following semi-discrete scheme:

$$\frac{W^k - W^{k-1}}{\tau} + \frac{\partial \psi^{k-1}}{\partial x} \frac{\partial W^k}{\partial y} - \frac{\partial \psi^{k-1}}{\partial y} \frac{\partial W^k}{\partial x} - \nu \frac{\partial W^k}{\partial x} - \alpha \psi^{k-1} \frac{\partial \psi^k}{\partial x} = 0. \tag{4.1}$$

We assume that $W(t,x,y)$ and $\psi(t,x,y)$ are sufficiently smooth functions. Equation (4.1) approximates equation (2.2) at the point (x,y,t_k) with accuracy $O(\tau)$.

Let us cover area Ω by a grid and denote by h_1 the grid spacing in the x-direction and by h_2- in the y-direction:

$$h_1 = \frac{2a_1}{N_1}, \quad h_2 = \frac{2a_2}{N_2},$$

where $N_1\ (>1)$ and $N_2\ (>1)$ are the number of grid points in the x and y directions respectively.

If in equation (4.1) we replace the first order derivatives with respect to space variables by central differences, we obtain the following difference equation:

$$\frac{W_{i,j}^{k}-W_{i,j}^{k-1}}{\tau}+\frac{\psi_{i+1,j}^{k-1}-\psi_{i-1,j}^{k-1}}{2h_1}\frac{W_{i,j+1}^{k}-W_{i,j-1}^{k}}{2h_2}-\frac{\psi_{i,j+1}^{k-1}-\psi_{i,j-1}^{k-1}}{2h_2}\frac{W_{i+1,j}^{k}-W_{i-1,j}^{k}}{2h_1}$$
$$-v\frac{W_{i+1,j}^{k}-W_{i-1,j}^{k}}{2h_1}-\alpha\psi^{k-1}\frac{\psi_{i+1,j}^{k}-\psi_{i-1,j}^{k}}{2h_1}=0,\quad i=2,...,N_1-1,\quad j=2,...,N_2-1. \tag{4.2}$$

It is obvious that difference equation (4.2) approximates equation (2.2) with accuracy $O\left(\tau+h_1^2+h_2^2\right)$ at the point $\left(x_i,y_j,t_k\right)$.

Reconstruction of the stream function field by means of vortex field can be accomplished from the following standard difference equation corresponding to equation (2.3):

$$\frac{\psi_{i+1,j}^{k}-2\psi_{i,j}^{k}+\psi_{i-1,j}^{k}}{h_1^2}+\frac{\psi_{i,j+1}^{k}-2\psi_{i,j}^{k}+\psi_{i,j-1}^{k}}{h_1^2}-\gamma\psi_{i,j}^{k}=W_{i,j}^{k}-\beta y_j, \tag{4.3}$$

where $i=2,...,N_1-1,\quad j=2,...,N_2-1.$

The difference equation (4.3) approximates equation (2.3) with accuracy $O\left(h_1^2+h_2^2\right)$ at the point $\left(x_i,y_j,\cdot\right)$.

We choose the initial conditions for the system of difference equations (4.2) and (4.3) according to the function $\psi_0(x,y)$, while the boundary conditions are chosen following from the consistency constraint according to the initial conditions.

We solve the system of difference equations (4.2), (4.3) by the following iteration (In order to simplify writing, we omit the index k in $W_{i,j}^{k}$ and $\psi_{i,j}^{k}$):

$$\overset{n}{W}_{i,j}=\frac{1}{2}\alpha_1\left(\delta_y\psi_{i,j}^{k-1}+v\right)\left(\overset{n-1}{W}_{i+1.j}-\overset{n-1}{W}_{i-1.j}\right)-\frac{1}{2}\alpha_2\delta_x\psi_{i,j}^{k-1}\left(\overset{n-1}{W}_{i.j+1}-\overset{n-1}{W}_{i.j-1}\right)$$
$$+\frac{1}{2}\alpha\alpha_1\psi_{i,j}^{k-1}\left(\overset{n-1}{\psi}_{+1i.j}-\overset{n-1}{\psi}_{-1i.j}\right)+W_{i,j}^{k-1}, \tag{4.4}$$

$$\frac{\overset{n}{\psi}_{i+1,j}-2\overset{n}{\psi}_{i,j}+\overset{n}{\psi}_{i-1,j}}{h_1^2}+\frac{\overset{n}{\psi}_{i,j+1}-2\overset{n}{\psi}_{i,j}+\overset{n}{\psi}_{i,j-1}}{h_1^2}-\gamma\psi_{i,j}=\overset{n}{W}_{i,j}-\beta y_j, \tag{4.5}$$

where n is an iteration index $(n=1,2,...)$, $\alpha_1=\dfrac{\tau}{h_1}$, $\alpha_2=\dfrac{\tau}{h_2}$, δ_x is a central difference operator by x, and δ_y is a same operator by y.

The transition step τ from one time level to the next is subject to the condition:

$$q=a_1\max_{i,j}\left|\delta_y\psi_{i,j}^{k-1}+v\right|+a_2\max_{i,j}\left|\delta_x\psi_{i,j}^{k-1}\right|+\frac{1}{\gamma}va_1\max_{i,j}\left|\psi_{i,j}^{k-1}\right|<1,\quad i=1,...,N_1,\quad j=1,...,N_2,$$

which represents a sufficient condition of convergence of iteration process (4.4), (4.5).

We solve the system of difference equations (4.3) with respect to the variable x by means of factorization method, and with respect to the variable y - by means of iteration (In order to simplify writing we omit the index k in $\psi_{i,j}^{k}$)

$$-\overset{n}{\psi}_{i+1,j}+a\overset{n}{\psi}_{i,j}-\overset{n}{\psi}_{i-1,j}=a_0^2\left(\overset{n-1}{\psi}_{i,j+1}-\overset{n}{\psi}_{i,j-1}\right)+h_1^2\left(W_{i,j}^k-\beta y_j\right), \tag{4.6}$$

where $i=2,...,N_1-1$ and $j=2,...,N_2-1$; n is an iteration index $(n=1,2,...)$, $a_0=\dfrac{h_1}{h_2}$ and $a=2+2a_0^2+\gamma h_1^2$.

Let us make the following remark:

Remark 2 : *The convergence rate of the iteration process (4.6) is high, which is conditioned by the fact that the diagonal domination of matrix of the system is significantly high. Another important factor is that in the iterations (4.4) and (4.6) we take the value of the streamfunction at the preceding time level as a first approximation.*

4. Results of numerical experiment and their analysis

On the basis of scheme (4.2),(4.3), numerical calculations are accomplished for the following values of parameters: $r_0=0.837$, $\gamma=0.6$, $\beta=0.4$ and $a=\pm 1$. Here we took the Rossby solitary wave as the initial condition (see (2.4) formula) and observed its evolution. The action of scalar nonlinearity is notable from $t=5$ and gradually intensifies. this action manifests itself in destruction of the initial dipole structure and in tending to form only monopole structure. At $t=7$ (more obviously at moment $t=10$, see Fig.1) it can be already seen the following scene: Anticyclone or cyclone dominates with much more advantage according to whether we take the scalar nonlinearity with plus or minus (We consider both cases, see Fig.2). Fig.2 b)) corresponds to the case $a=1$, in this case dominates anticyclone, while Fig.2. a)) corresponds to the case $a=-1$, and in this case dominates a cyclone. It is clearly seen that Fig.2 a)) and Fig.2 b)) are symmetric to each other. It can be obviously seen from the numerical experiment that the wave moves, and for this reason, in order to catch the process of intensifying of anticyclone and cyclone, it is necessary to take a domain of integration as large as possible. In one case we took $a_1=5$ (see Fig.1), while in the other case $a_1=10$ (see Fig.2). In the other case, even for such rough values of parameters as $h=0.1$ and $\tau=0.01$, an action of scalar nonlinearity can be obviously seen at moment $t=10$. Let us note that, for convergence of iterative process, it is necessary to keep a certain relation between the steps τ and h.

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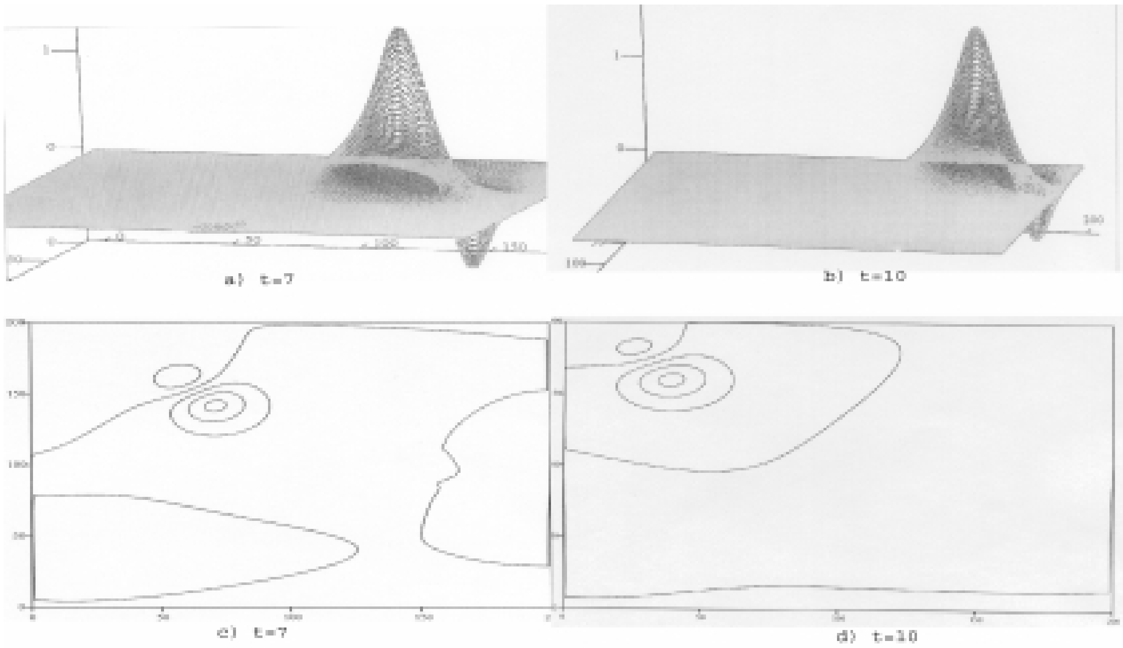
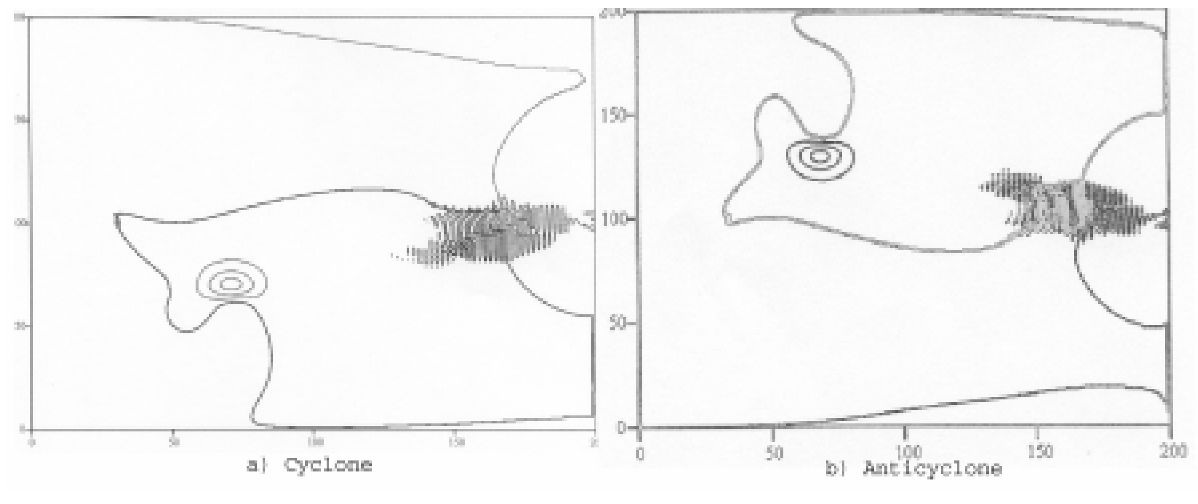


$Fig.1.\quad h = 0.05,\quad \tau = 0.00625,\quad a_1 = a_2 = 5,\quad a = 1.$



$Fig.2.\quad h = 0.1,\quad \tau = 0.01,\quad a_1 = a_2 = 10,\quad t = 10,\quad In\ case\ a))\ a = -1;\quad In\ case\ b))\ a = 1.$

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