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ON STABILITY AND CONVERGENCE OF THE FOURTH LAYER IMPLICIT SEMI-DISCRETE SCHEME FOR ABSTRACT PARABOLIC EQUATION

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Abstract

In the present work there is considered the four-layer implicit semi-discrete scheme for an abstract parabolic equation. There is obtained an explicit presentation for the solution of the corresponding discrete problem using operator polynomials. On the basis of this presentation, there is obtained a priori estimates in natural classes for the solution of the discrete problem, when the spatial operator is self-adjoint and positive definite. From these estimates follow the stability and convergence of the considered scheme.

Introduction

In the present work for the investigation of the multi layer semi-discrete schemes we are guided by the following conception: To each semi-discrete scheme, which gives an approximate solution of an evolution problem, there correspond polynomials of certain classes. These polynomials we call as polynomials associated to the scheme.

In case of equation with constant operator, the corresponding semi-discrete scheme generates polynomials, which represent a generalization of Chebischev polynomials of the second kind. According to this polynomials there is constructed the implicit representation of the difference problem and on the basis of this polynomials there is made a conclusion about stability of the difference scheme. The a priori estimates also are obtained ([1],[2],[3]).

To the investigation of multi-layer semi-discrete schemes for an abstract parabolic equation there are dedicated works of M. Crouzeix [4], M. Crouzeix, A. Raviart [5], M.N. Le Roux [6], M. Zlamal [7], [8].

In the work of M. Crouzeix, A. Raviart [5] attention is paid to the fact that stability of the abstract scheme with maximum accretive operator A follows from the stability of the corresponding scalar scheme with parameter Z, with $\mathrm{Re}(Z) \geq 0$.

Results, presented in this work, are included in the Ph.D. dissertation paper of one of the coauthors (see [9]) and it is written in Georgian.

1. Statement of the problem and main results

Let us consider Cauchy abstract problem in Hilbert space H

$$u'(t) + Au(t) = f(t), \quad t > 0, \quad u(0) = \varphi, \tag{1.1}$$

where A is a selfadjoint positive definite operator with definition domain D(A); f(t) is an abstract function with values from H; $\varphi \in H$, u(t) is a searched function.

Let $f(t)$ be a continuous function by Gelder in $t \ge 0$. As it is clear that the operator $(-A)$ is a generator of the analytic semi-group $\exp(-tA)$, the function $u(t)$ is defined by the following formula for any $\varphi \in H$ (see for example К. Като [10] p. 609)

$$u(t) = U(t)\varphi + \int_0^t U(t-s) f(s) ds, \tag{1.2}$$

where $U(t) = \exp(-tA)$ is continuous when $t \ge 0$, continuously differentiable at $t > 0$ and represents the solution of problem (1.1).

We search the solution of problem (1.1) by the following four-layer semi-discrete scheme:

$$\frac{\frac{11}{6}u_{k+1} - 3u_k + \frac{3}{2}u_{k-1} - \frac{1}{3}u_{k-2}}{\tau} + Au_{k+1} = f(t_{k+1}), \quad k = 2,3,..., \tag{1.3}$$

where $\tau > 0$ is a step of the mesh according to t, u_k is an approximate solution of problem (1.1) at point $t = t_k = k\tau$; u_0, u_1, u_2 - are given initial vectors from H.

The following theorem is valid:

Theorem: *For scheme (1.3), the following a priori estimates are valid:*

$$\|u_{k+2}\| \le c\left[\|u_0\| + \|u_1\| + \|u_2\| + \tau\sum_{i=1}^{k}\|f(t_{i+2})\|\right], \tag{1.4}$$

$$\|A^\alpha u_{k+2}\| \le c\left(\frac{11\alpha}{2}\right)^\alpha \left[\frac{1}{t_{k+3}^\alpha}\|u_2\| + \left(\frac{1}{t_{k+2}^\alpha} + \frac{1}{t_{k+1}^\alpha}\right)\|u_1\| + \frac{1}{t_{k+2}^\alpha}\|u_0\| + \tau\sum_{i=1}^{k}\frac{1}{t_{k+3-i}^\alpha}\|f(t_{i+2})\|\right], \tag{1.5}$$

where $0 < \alpha < 1$, $c = const > 0$, $\|\cdot\|$ *- is a norm in H.*

Proof: Let us define u_{k+1} from (1.3), we obtain:

$$u_{k+1} = \frac{18}{11}Lu_k - \frac{9}{11}Lu_{k-1} + \frac{2}{11}Lu_{k-2} + \frac{6}{11}\tau Lf(t_{k+1}), \tag{2.1}$$

where

$$L = \left(I + \frac{6}{11}\tau A\right)^{-1}.$$

Let us introduce the following denotations:

$$L_1 = \frac{18}{11}L, \quad L_2 = -\frac{9}{11}L, \quad L_3 = \frac{2}{11}L, \quad g_{k+1} = \frac{6}{11}\tau Lf(t_{k+1}),$$

then (2.1) can be written as follows:

$$u_{k+1} = L_1u_k + L_2u_{k-1} + L_3u_{k-2} + g_{k+1}.$$

from here, by induction, we obtain (see. [3], p. 59):

$$u_{k+2}=L_1U_ku_2+(L_2U_{k-1}+L_3U_{k-2})u_1+L_3U_{k-1}u_0+\sum_{i=1}^{k}U_{k-i}g_{i+2}, \tag{2.2.}$$

where operator polynomials $U_k(L_1,L_2,L_3)$ are defined by the following recurrent relation:

$$U_k(L_1,L_2,L_3)=L_1U_{k-1}+L_2U_{k-2}+L_3U_{k-3}, \tag{2.3}$$
$$U_0=I,\quad U_{-1}=U_{-2}=0.$$

Let us apply the operator A^{α} $(0<\alpha<1)$ to both sides of equality (2.2) and pass to norms, we obtain:

$$\begin{aligned}\left\|A^{\alpha}u_{k+2}\right\|\le\frac{18}{11}\Big(\left\|A^{\alpha}LU_k\right\|\left\|u_2\right\|+\Big[\left\|A^{\alpha}LU_{k-1}\right\|+\left\|A^{\alpha}LU_{k-2}\right\|\Big]\left\|u_1\right\|+\\ +\left\|A^{\alpha}LU_{k-1}\right\|\left\|u_0\right\|+\tau\sum_{i=1}^{k}\left\|A^{\alpha}LU_{k-1}\right\|\left\|f(t_{i+2})\right\|\Big).\end{aligned} \tag{2.4}$$

Let us estimate $\left\|A^{\alpha}LU_k\right\|$. For this purpose we need the following formula (see. [3], p. 66):

$$U_k(sx_1,s^2x_2,s^3x_3)=s^kU_k(x_1,x_2,x_3)\,.$$

According to this formula we obtain:

$$\left\|A^{\alpha}LU_k\right\|=\left\|A^{\alpha}LL^{\frac{k}{3}}U_k\left(\frac{18}{11}L^{\frac{2}{3}},-\frac{9}{11}L^{\frac{1}{3}},\frac{2}{11}I\right)\right\|\le\left\|A^{\alpha}LL^{\frac{k}{3}}\right\|\left\|U_k\left(\frac{18}{11}L^{\frac{2}{3}},-\frac{9}{11}L^{\frac{1}{3}},\frac{2}{11}I\right)\right\|. \tag{2.5}$$

As well-known, the norm of operator-polynomial, when argument is a self-adjoint bounded operator, is equal to C-norm of the corresponding scalar polynomial on spectrum (see [11], p. 346). Due to this result we have:

$$\left\|U_k\left(\frac{18}{11}L^{\frac{2}{3}},-\frac{9}{11}L^{\frac{1}{3}},\frac{2}{11}I\right)\right\|\le\max_{x\in[0,1]}\left|U_k\left(\frac{18}{11}x^2,-\frac{9}{11}x,\frac{2}{11}\right)\right|. \tag{2.6}$$

Here and above we have used the fact that $Sp(L)\in[0,1]$.

Obviously, according to (2.3), polynomials

$$P_k(x)=U_k\left(\frac{18}{11}x^2,-\frac{9}{11}x,\frac{2}{11}\right)$$

satisfy the following recurrent relation:

$$P_k(x)=\frac{18}{11}x^2P_{k-1}-\frac{9}{11}xP_{k-2}+\frac{2}{11}P_{k-3}, \tag{2.7}$$
$$P_0\equiv1,\ P_{-1}\equiv P_{-2}\equiv0.$$

The characteristic equation, corresponding to difference equation (2.7), is of the following form:

$$Q(\lambda)=\lambda^3-\frac{18}{11}x^2\lambda^2+\frac{9}{11}x\lambda-\frac{2}{11}=0,\qquad x\in[0,1]. \tag{2.8}$$

Let us show that there exists $\gamma \in \left]1, \frac{11}{4}\right[$ such that, for any $x \in [0,1[$ the real root of the characteristic equation (2.8) is placed in the interval $\left]\frac{2}{11}\gamma, 1\right[$.

Indeed,

$$Q(\frac{2}{11}\gamma) = -9\gamma^2(15 \cdot 11^2 - 64\gamma^3) < 0,$$

$$Q(1) = -\frac{9}{11}(2x^2 - x - 1) > 0,$$

for any $x \in [0,1[$. Therefore the real root of the characteristic equation (2.8) $\lambda_1 \in \left[\frac{2}{11}\gamma, 1\right]$.

Below we will show that other two roots λ_2 and λ_3 of the equation (2.8) are complex, $\lambda_3 = \overline{\lambda_2}$. Then, from (2.8) according to Vieta theorem, we have:

$$\left|\lambda_1 \cdot \lambda_2 \cdot \overline{\lambda_2}\right| = \frac{2}{11}.$$

Hence

$$\left|\lambda_2\right|^2 = \frac{2}{11\lambda_1} < \frac{1}{\gamma} < 1.$$

Thus, absolute values of complex roots of the characteristic equation (2.8), for any $x \in [0,1]$, are not more than a number less than one, which does not depend on x, and the real root belongs to unit circle. From here it follows that polynomials $P_k(x)$ are uniformly bounded in $[0,1]$.

From this fact and (2.6), we can obtain:

$$\left\| U_k\left(\frac{18}{11}L^{\frac{2}{3}}, -\frac{9}{11}L^{\frac{1}{3}}, \frac{2}{11}I\right)\right\| \le c, \quad c = const > 0. \qquad (2.9)$$

It is left to show that the characteristic equation (2.8) has complex roots.

As well-known, a cubic equation has complex roots, if its discriminant is less than zero. In our case the discriminant of the equation (2.8) is:

$$D = -108\left(\frac{q^2(x)}{4} + \frac{p^3(x)}{27}\right),$$

where

$$p(x) = \frac{9}{11}x(1 - x^3) - \frac{9}{11^2}x^4,$$

$$q(x) = \frac{2^4 \cdot 3^3}{11^3}x^6 + \frac{2 \cdot 3^3}{11^2}x^3 - \frac{2}{11}.$$

For any $x \in [0,1]$, the equalities are valid:

$$p(x) \ge -\frac{9}{11^2}, \; q(x) \le -\frac{5}{11 \cdot 2^4}.$$

According to these estimates, we have:

$$D=-108\left(\frac{q^2(x)}{4}+\frac{p^3(x)}{27}\right)\le -108\frac{1}{11^2}\left(\frac{5^2}{2^{10}}-\frac{3^3}{11^4}\right).<0.$$

Let us return to the equality (2.5). According to the estimate (2.9) we have:

$$\left\|A^{\alpha}LU_k\right\|\le c\left\|A^{\alpha}LL^{\frac{k}{3}}\right\|,\qquad c=const>0.$$

Since

$$A^{\alpha}LL^{\frac{k}{3}}=A^{\alpha}L^{\alpha}L^{\frac{k-3\alpha+3}{3}}=\left(\frac{11}{6}\right)^{\alpha}\frac{1}{\tau^{\alpha}}(I-L)^{\alpha}L^{\frac{k-3\alpha+3}{3}},$$

therefore

$$\left\|A^{\alpha}LL^{\frac{k}{3}}\right\|\le\left(\frac{11}{6}\right)^{\alpha}\frac{1}{\tau^{\alpha}}\max_{x\in[0,1]}\left|(1-x)^{\alpha}x^{\frac{k-3\alpha+3}{3}}\right|.$$

Let us estimate the function

$$\varphi(x)=(1-x)^{\alpha}x^{\frac{k-3\alpha+3}{3}},\ x\in[0,1].$$

Since the critical point of this function is

$$x_0=\frac{k-3\alpha+3}{k+3},$$

therefore we have:

$$\varphi(x)\le\varphi(x_0)\le\left(1-\frac{k-3\alpha+3}{k+3}\right)^{\alpha}=\left(\frac{3\alpha}{k+3}\right)^{\alpha}.$$

Thus, the following estimate is valid:

$$\left\|A^{\alpha}LU_k\right\|\le c\left(\frac{11\alpha}{2}\right)^{\alpha}\frac{1}{t_{k+3}^{\alpha}}. \tag{2.10}$$

From (2.4), taking into account (2.10) the a priori estimate (1.5) is obtained.

The estimate (1.4) is proved analogously.

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