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ON ONE BOUNDARY VALUE PROBLEM FOR THE NON-SHALLOW SPHERICAL SHELL

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It is known that the equation of balance and stress-strain relations (Hook's Law) has the following form in any system of curve linear coordinates [1]

$$\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}\vec{\sigma}^i}{\partial x^i}+\vec{\phi}=0\,, \tag{1}$$

$$\vec{\sigma}^i=\lambda\left(\vec{R}^k\partial_k\vec{u}\right)\vec{R}^i+\mu\left[\left(\vec{R}^i\partial_k\vec{u}\right)\vec{R}^k+\left(\vec{R}^iR^k\right)\partial_k\vec{u}\right]\,, \tag{2}$$

where g is the discriminant of the metric tensor of the space, $\vec{\sigma}^i$ are contra variant stress vectors, $\vec{\phi}$ is an external force, $\vec{R}^i$ are contra variant base vectors of the space, $\vec{u}$ is the displacement vector, λ and μ are Lame's constants.

We introduce at the sphere the isometric coordinates system, where

$$x_1=tg\frac{\theta}{2}\cos\varphi\,,\quad x_2=tg\frac{\theta}{2}\sin\varphi$$

are the isometric coordinates on the shell mid surface, with respect to which the metric quadratic form is

$$ds^2=\Lambda\left(x^1,x^2\right)\left[(dx^1)^2+\left(dx^2\right)^2\right],\qquad \Lambda=\frac{4\rho^2}{\left(1+z\bar{z}\right)^2}\,.$$

Let suppose that the displacement vector is independent from the thickness coordinate x_3

$$\vec{u}\left(x^1,x^2,x^3\right)=\vec{u}\left(x^1,x^2\right).$$

Using the notation

$$\vec{\sigma}^\alpha\left(x^1,x^2,x^3\right)=\frac{1}{\left(1+\frac{x_3}{\rho}\right)^2}\vec{T}^\alpha\left(x^1,x^2\right),\qquad \vec{\sigma}^3\left(x^1,x^2,x^3\right)=\frac{1}{1+\frac{x^3}{\rho}}\vec{T}^\alpha\left(x^1,x^2\right),$$

$$\vec{\phi}\left(x^1,x^2,x^3\right)=\frac{1}{\left(1+\frac{x_3}{\rho}\right)^2}\vec{F}\left(x^1,x^2\right),$$

where ρ is a radius of sphere, we can rewrite complete system of equation of balance and Hook's Law as [3]

$$\frac{1}{\Lambda}\frac{\partial}{\partial z}\left(\mathrm{T}_{11}-\mathrm{T}_{22}+2i\mathrm{T}_{12}\right)+\frac{\partial}{\partial\bar{z}}\mathrm{T}_{\alpha}^{\alpha}+\frac{2}{\rho}\mathrm{T}_{+}+F_{+}=0,$$
$$\frac{1}{\Lambda}\left(\frac{\partial\mathrm{T}_{+}}{\partial z}+\frac{\partial\overline{\mathrm{T}}_{+}}{\partial\bar{z}}\right)+\frac{1}{\rho}\left(\mathrm{T}_{3}^{3}-\mathrm{T}_{\alpha}^{\alpha}\right)+F_{3}=0. \tag{3}$$

$$\begin{cases}\mathrm{T}_{11}-\mathrm{T}_{22}+2i\mathrm{T}_{12}=4\mu\Lambda\dfrac{\partial u^{+}}{\partial\bar{z}},\\ \mathrm{T}_{\alpha}^{\alpha}=2(\lambda+\mu)\left(\theta+\dfrac{2}{\rho}u_{3}\right),\quad \theta=\dfrac{1}{\Lambda}\left(\dfrac{\partial u_{+}}{\partial z}+\dfrac{\partial\bar{u}_{+}}{\partial\bar{z}}\right),\\ \mathrm{T}_{+}=\mathrm{T}_{13}+i\mathrm{T}_{23}=\mathrm{T}_{31}+i\mathrm{T}_{32}=\mu\left(2\dfrac{\partial u_{3}}{\partial\bar{z}}-\dfrac{1}{\rho}u_{+}\right),\\ \mathrm{T}_{33}=\lambda\left(\theta+\dfrac{2}{\rho}u_{3}\right)=\dfrac{\lambda}{2(\lambda+\mu)}\mathrm{T}_{\alpha}^{\alpha},\\ F_{+}=F_{1}+iF_{2},\qquad u_{+}=u_{1}+iu_{2,}\qquad u^{+}=u^{1}+iu^{2}.\end{cases} \tag{4}$$

The general representations of this system are given in the form:

$$u_{+}=2\rho\left[\frac{\partial\chi}{\partial\bar{z}}-\frac{z}{1+z\bar{z}}\overline{\varphi}[z]-\overline{\psi}(z)\right],$$
$$u_{3}=\chi(z,\bar{z})+\frac{\lambda+3\mu}{4(\lambda+2\mu)}\left[\varphi(z)+\overline{\varphi}(z)\right],$$

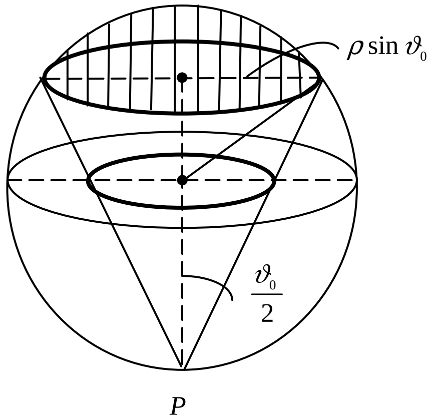
where $\varphi(z)$ and $\psi(z)$ are holomorphic functions of z and $\chi(z,\bar{z})$ is solution of equation $\nabla^{2}\chi+\frac{2}{\rho^{2}}\chi=0$, which is expressed with the help of holomorphic functions $f(z)$ by formula

$$\chi(z,\bar{z})=2\,\mathrm{Re}\left[f(z)-\frac{2\bar{z}}{1+z\bar{z}}\int_{0}^{z}f(t)dt\right].$$

The combinations of the stress vectors are represented as follows [4]

$$\begin{cases}\mathrm{T}_{11}-\mathrm{T}_{22}+2i\mathrm{T}_{12}=8\mu\rho\left[\overline{f''}(z)-\dfrac{z}{1+z\bar{z}}\left(\overline{\varphi'}(z)+\dfrac{z}{1+z\bar{z}}\overline{\varphi}(z)\right)-\left(\overline{\psi'}(z)+\dfrac{2z}{1+z\bar{z}}\overline{\psi}(z)\right)\right],\\ \mathrm{T}_{\alpha}^{\alpha}=\dfrac{\mu}{\rho}\dfrac{\lambda+\mu}{\lambda+2\mu}\left[\varphi(z)+\overline{\varphi}(z)\right],\\ \mathrm{T}_{3}^{3}=\dfrac{\lambda}{2(\lambda+\mu)}\mathrm{T}_{\alpha}^{\alpha},\\ T_{+}=\mu\left[\dfrac{1}{2}\dfrac{\lambda+3\mu}{\lambda+2\mu}\overline{\varphi'}(z)+\dfrac{2z}{1+z\bar{z}}\overline{\varphi}(z)+2\overline{\psi}(z)\right]\end{cases} .(5)$$

Let's consider the boundary value problem for the non-shallow spherical shell. We have to find the elasticity balance, when the displacements are marked on the boundary points.



The boundary conditions for the components of the displacement vector have the form

$$\begin{cases} u_{(l)} + iu_{(s)} = iu_+ \dfrac{d\bar{z}}{ds} = 2\rho i \left[\dfrac{\partial \chi}{\partial \bar{z}} - \dfrac{z}{1+z\bar{z}} \overline{\varphi}(z) - \overline{\psi}(z) \right] \dfrac{d\bar{z}}{ds}, \\ u_3 = \chi(z,\bar{z}) + \dfrac{\lambda + 3\mu}{4(\lambda + 2\mu)} [\varphi(z) + \overline{\varphi}(z)]. \end{cases}$$

Let's consider that the boundary conditions have the form

$$\begin{cases} u_{(l)} + iu_{(s)} = \sum\limits_{n=-\infty}^{+\infty} (A_n + iB_n) e^{in\varphi}, \quad {}_{r=r_0}, \\ u_3 = \sum\limits_{n=-\infty}^{+\infty} M_n \, e^{in\varphi}, \quad {}_{r=r_0}, \end{cases}$$

(6)

where $A_{-n} = \overline{A}_n$, $B_{-n} = \overline{B}_n$, $M_{-n} = \overline{M}_n$.

If the functions $\varphi(z)$, $\psi(z)$, $f(z)$ are introduced by series

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad \psi(z) = \sum_{n=0}^{\infty} b_n z^n, \qquad f(z) = \sum_{n=0}^{\infty} c_n z^n,$$

then from (6) we'll have the following system

$$\begin{aligned} u_{(l)} + iu_{(s)} = \sum_{n=0}^{\infty} \left(1 + r_0^2\right) \Bigg\{ & \frac{r_0^{n+1}}{n+1} \left[1 + \frac{2}{(n+2)(n+3)} \frac{r_0^4}{\left(1+r_0^2\right)^2} - \frac{2}{n+2} \frac{r_0^2}{1+r_0^2} \right] \bar{c}_n e^{-i(n+2)\varphi} - \\ & - \frac{2}{(n+1)(n+2)(n+3)} \frac{r_0^{n+3}}{\left(1+r_0^2\right)^2} c_n e^{i(n+2)\varphi} - \frac{r_0^{n+1}}{1+r_0^2} \bar{a}_n e^{-in\varphi} - r_0^n \bar{b}_n e^{-i(n+1)\varphi} \Bigg\} = \\ = \sum_{n=-\infty}^{+\infty} (A_n + iB_n) e^{in\varphi}, \qquad & r = r_0. \end{aligned} \qquad (7)$$

$$u_3=\sum_{n=0}^{\infty}\left\{\frac{n+3+(n+1)r_0^2}{(n+1)(n+2)(n+3)}\frac{r_0^{n+2}}{1+r_0^2}\left(c_n e^{i(n+2)\varphi}+\bar{c}_n e^{-i(n+2)\varphi}\right)+\right.$$
$$\left.+\frac{\lambda+3\mu}{4(\lambda+2\mu)}r_0^n\left(a_n e^{in\varphi}+a_n e^{-in\varphi}\right)\right\}=\sum_{n=-\infty}^{+\infty}M_n e^{in\varphi},\qquad r=r_0. \tag{8}$$

The solutions of this system (7)-(8) have the following form:

$$a_0=-\frac{1}{r_0}\left(\bar{A}_0-i\bar{B}_0\right),$$

$$a_1=\frac{1}{r_0}\frac{4(\lambda+2\mu)}{\lambda+3\mu}M_1,$$

$$a_2=\frac{1}{r_0^2}\frac{4(\lambda+2\mu)}{\lambda+3\mu}\left[\frac{3+r_0^2}{2r_0}(A_2+iB_2)+M_2\right],$$

$$a_n=\frac{1}{r_0^n}\frac{4(\lambda+2\mu)}{\lambda+3\mu}\left[M_n+\frac{n+1+(n-1)r_0^2}{2r_0}(A_n+iB_n)\right],\qquad n\geq 3,$$

$$b_0=\frac{1}{1+r_0^2}\left[iB_1-A_1-r_0\frac{4(\lambda+2\mu)}{\lambda+3\mu}M_1\right],$$

$$b_1=-\frac{1}{1+r_0^2}\left\{\left[\frac{3+3r_0^2+r_0^4}{r_0^3}+\frac{2(\lambda+2\mu)}{\lambda+3\mu}\frac{3+r_0^2}{r_0}\right](A_2+iB_2)+\frac{4(\lambda+2\mu)}{\lambda+3\mu}M_2+\frac{1}{r_0}(A_2-iB_2)\right\},$$

$$b_n=-\left\{\frac{1+r_0^2}{2r_0^{n+2}}(n+1)(n+2)+\frac{r_0^{2-n}}{1+r_0^2}-\frac{n+2}{r_0^n}+\frac{2(\lambda+2\mu)}{\lambda+3\mu}\frac{n+2+nr_0^2}{r_0\left(1+r_0^2\right)}\right\}(A_{n+1}+iB_{n+1})-$$
$$-\frac{1}{r_0^n\left(1+r_0^2\right)}(A_{n+1}-iB_{n+1})-\frac{4(\lambda+2\mu)}{\lambda+3\mu}\frac{1}{r_0^{n-1}\left(1+r_0^2\right)}M_{n+1},\qquad n\geq 2,$$

$$c_0=-\frac{3\left(1+r_0^2\right)}{r_0^3}[A_2+iB_2],$$

$$c_n=-\left(n^3+6n^2+11n+6\right)\frac{1+r_0^2}{2r_0^{n+3}}(A_{n+2}+iB_{n+2}),\qquad n\geq 1,$$

where

$$M_0=-\frac{\lambda+3\mu}{\lambda+2\mu}\frac{1}{2r_0}(A_0+iB_0).$$

Therefore, it solves three-dimensional problem for the non-shallow spherical shell for the components of the displacement vector, when the displacement vector is independent from the thickness coordinate x_3.

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