A METHOD OF SOLUTION OF AN INITIAL AND TIME DEPENDENT BOUNDARY VALUE PROBLEM FOR A DYNAMIC STRING NONLINEAR DIFFERENTIAL EQUATION

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(Received 07.01.2024; accepted 10.06.2024)

Abstract

An initial boundary value problem for Kirchhoff string nonlinear differential equation is considered in case of time dependent inhomogeneous boundary condition of the first type. The problem is reduced to a solution of more complex equation than the original, the types of the initial and boundary conditions remained as before, but now the boundary condition has become homogeneous. To find an approximate solution of the received problem, a numerical algorithm that is a combination of the projection method, the implicit difference scheme, and the iteration process is used.

Keywords and phrases: Kirchhoff string equation, inhomogeneous boundary conditions, zeroing of boundary conditions, Galerkin method, implicit difference scheme, Jacobi–Cardano iteration.

AMS subject classification (2010): 35L20, 65H10, 65M60, 65N06, 74G15.

1 Statement of Problem

Let us consider the nonlinear equation

$$w_{tt}(x,t) - \left(\alpha + \beta \int_0^L w_x^2(x,t)\,dx\right) w_{xx}(x,t) = f(x,t), \qquad 0 < x < L, \quad 0 < t \le T, \tag{1}$$

with initial boundary conditions

$$\begin{gathered} w(x,0) = w^0(x), \quad w_t(x,0) = w^1(x), \\ w(0,t) = \mu(t), \quad w(L,t) = \nu(t), \\ 0 \le x \le L, \quad 0 \le t \le T. \end{gathered} \tag{2}$$

Here, $\alpha > 0$, $\beta > 0$, L and T are the given constants, while $f(x,t) \in L_2((0,L) \times (0,T])$, $w^0(x) \in W_2^2[0,L]$, $w^1(x) \in L_2[0,L]$, $\mu(t), \nu(t) \in C_2[0,T]$ are the given functions.

In 1876, whcn studying the vibration of a string, a German physicist G. Kirchhoff [16] obtained an integro-differential equation of the type (1) in case $f(x,t) = 0$. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations.

Equation (1) and its generalizations have attracted the attention of manu researchers. Problem of solvability was investigated by Arosio and Garavaldi [1], Bernstein [3], Cavalcanti et al. [5], D'Ancona and Spagnolo [8], Greenberg and Hu [10], Hirosawa[11, 12], Izaguerre et al. [13], Lapa [18], Manfrin [19], Medeiros [21], Newman [22], Nishihara [24], Pokhoshaev [27] and others. The problem of methods of approximate solution, which is considered in this paper, was investigated by Attigue [2], Bilbao [4], Chandhary and Srivastava [6], Christie and Sanz-Serna [7], Dickey [9], Kachakhidze et al. [15], Mbehou et al. [20], Ngoc et al. [23], Peradze [25], Peradze et al. [26], Rincon and Rodrigues [28], Rincon et al. [29], Rogava and Vashakidze [30], Truong et al. [31].

The approximate methods of solution of some class of parabolic integro-differential equations are investigated by Jangveladze et al. [14].

2 Reducing Inhomogeneous Boundary Conditions to Homogeneous Ones

Let us introduce an auxiliary function $u(x,t)$, using the formula

$$w(x,t) = u(x,t) + \left(\frac{L-x}{L}\,\mu(t) + \frac{x}{L}\,\nu(t)\right), \quad 0 \le x \le L, \quad 0 \le t \le T. \tag{3}$$

Therefore,

$$u(x,t) = w(x,t) - \left(\frac{L-x}{L}\,\mu(t) + \frac{x}{L}\,\nu(t)\right). \tag{4}$$

Define the task for new function. Using (1)–(4), we get

$$
\begin{gathered}
w_{tt}(x,t) = u_{tt}(x,t) + \left(\frac{L-x}{L}\,\mu''(t) + \frac{x}{L}\,\nu''(t)\right),\\
w_x(x,t) = u_x(x,t) + \frac{1}{L}(-\mu(t)+\nu(t)), \quad w_{xx}(x,t) = u_{xx}(x,t),\\
u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x),\\
u^0(x) = w^0(x) - \left(\frac{L-x}{L}\,\mu(0) + \frac{x}{L}\,\nu(0)\right),\\
u^1(x) = w^1(x) - \left(\frac{L-x}{L}\,\mu'(0) + \frac{x}{L}\,\nu'(0)\right),\\
u(0,t) = w(0,t) - (\mu(t)+0) = \mu(t) - \mu(t) = 0,\\
u(L,t) = w(L,t) - (0+\nu(t)) = \nu(t) - \nu(t) = 0.
\end{gathered} \tag{5}
$$

Applying (5), we get the following boundary value problem for function $u(x,t)$

$$
\begin{gathered}
u_{tt}(x,t) - \left\{\alpha + \beta\left[\int_0^L u_x^2(x,t)\,dx + \frac{2}{L}\big(-\mu(t)+\nu(t)\big)\int_0^L u_x(x,t)\,dx\right.\right.\\
\left.\left. + \frac{1}{L}\big(-\mu(t)+\nu(t)\big)^2\right]\right\} u_{xx}(x,t) = \varphi(x,t), \qquad (6)\\
0 < x < L, \quad 0 < t \le T,
\end{gathered}
$$

$$
\begin{gathered}
u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x),\\
u(0,t) = 0, \quad u(L,t) = 0,\\
0 \le x \le L, \quad 0 \le t \le T,
\end{gathered} \tag{7}
$$

where

$$
\varphi(x,t) = f(x,t) - \frac{1}{L}\big((L-x)\mu''(t) + x\nu''(t)\big). \tag{8}
$$

3 Numerical Algorithm for Auxiliary Problem

The algorithm of solution of problem (6), (7) contains three parts.

3.1. Space discretization-projection method

We write an approximate solution of problem (6), (7) in the form

$$
u_n(x,t) = \sum_{i=1}^{n} u_{ni}(t)\sin\frac{i\pi}{L}\,x, \tag{9}
$$

where the coefficients $u_{ni}(t)$ are defined by Galerkin's method from the system of ordinary differential equations and the conditions

$$u''_{ni}(t)+\left(\alpha+\frac{\beta L}{2}\sum_{j=1}^{n}\Big(\frac{j\pi}{L}\Big)^2u^2_{nj}(t)+\frac{\beta}{L}\big(-\mu(t)+\nu(t)\big)^2\right)$$
$$\times\Big(\frac{i\pi}{L}\Big)^2u_{ni}(t)=\psi_{ni}(t), \tag{10}$$
$$u_{ni}(0)=u_i^0,\quad u'_{ni}(0)=u_i^1,\quad i=1,2,\dots,n, \tag{11}$$

where based on (7) and (8),

$$\psi_{ni}(t)=\frac{2}{L}\int_0^L f(x,t)\sin\frac{i\pi}{L}x\,dx-\frac{2}{i\pi}\big(\mu''(t)+(-1)^{i+1}\nu''(t)\big),$$
$$i=1,2,\dots,n,$$
$$u_i^p=\frac{2}{L}\int_0^L u^p(x)\sin\frac{i\pi}{L}x\,dx,\quad p=0,1,\ \ i=1,2,\dots,n.$$

3.2. Time discretization-implicit difference scheme

To solve problem (10), (11) on the time interval $[0,T]$, we introduce the grid with the step $\tau=\frac{T}{M}$, $M>2$, and nodes $t_m=m\tau$, $m=0,1,\dots,M$. An approximate value of $u_{ni}(t)$ on the m-th level, i.e., for $t=t_m$, $m=0,1,\dots,M$, is denoted by u^m_{ni}. Firther, we need the notation $\psi^m_{ni}=\psi_{ni}(t_m)$, $m=0,1,\dots,M$, $i=1,2,\dots,n$.

We use the following difference scheme of Crank–Nicolson type

$$\frac{u^m_{ni}-2u^{m-1}_{ni}+u^{n-2}_m}{r^2}$$
$$+\frac{1}{2}\Big(\frac{i\pi}{L}\Big)^2\sum_{p=0}^{1}\left\{\left[\alpha+\frac{\beta L}{4}\sum_{j=1}^{n}\Big(\frac{j\pi}{L}\Big)^2\Big((u^{m-p}_{nj})^2+(u^{m-p-1}_{nj})^2\Big)\right.\right.$$
$$\left.\left.+\frac{\beta}{2L}\sum_{l=0}^{1}\big(-\mu(t_{m-p-l})+\nu(t_{m-p-l})\big)^2\right]\frac{u^{m-p}_{ni}+u^{m-p-1}_{ni}}{2}\right\}=\frac{1}{2}\big(\psi^m_{ni}+\psi^{m-2}_{ni}\big),$$
$$m=2,3,\dots,M,\quad i=1,2,\dots,n.$$

Let us represent this system of equations in the form

$$8\Big(\frac{L}{\tau i\pi}\Big)^2u^m_{ni}+\left[2\alpha+\frac{\beta L}{2}\sum_{j=1}^{n}\Big(\frac{j\pi}{L}\Big)^2\big((u^m_{nj})^2+(u^{m-1}_{nj})^2\big)\right.$$
$$\left.+\frac{\beta}{L}\sum_{l=0}^{1}\big(-\mu(t_{m-l})+\nu(t_{m-l})\big)^2\right](u^m_{ni}+u^{m-1}_{ni})=8\Big(\frac{L}{\tau i\pi}\Big)^2g^m_{ni}, \tag{12}$$
$$m=2,3,\dots,M,\quad i=1,2,\dots,n,$$

where

$$g_{ni}^m = 2u_{ni}^{m-1} - u_{ni}^{m-2} - \frac{\tau^2}{2}\Big(\alpha + \frac{\beta L}{2}\sum_{j=1}^{n}\Big(\frac{j\pi}{L}\Big)^2 \frac{(u_{nj}^{m-1})^2 + (u_{nj}^{m-2})^2}{2}\Big)$$
$$\times\Big(\frac{i\pi}{L}\Big)^2 \frac{u_{ni}^{m-1} + u_{ni}^{m-2}}{2} + \frac{\tau^2}{2}(\psi_{ni}^m + \psi_{ni}^{m-2}),$$
$$m = 2, 3, \dots, M, \quad i = 1, 2, \dots, n,$$

and using (3), (4), (6) and (11) supplement it with the equalities

$$\begin{aligned}
&u_{ni}^0 = u_i^0,\\
&u_{ni}^1 = u_i^0 + \tau u_i^1 + \frac{\tau^2}{L}\int_0^L \Big\{f(x,0) - \frac{1}{L}\big((L-x)\mu''(0) + x\nu''(0)\big)\\
&\quad + \Big[\alpha + \beta\int_0^L \Big(u^{0\prime}(x) + \frac{1}{L}\big(-\mu(0) + \nu(0)\big)\Big)^2 dx\Big]u^{0\prime\prime}(x)\Big\}\\
&\times \sin\frac{i\pi}{L}x\,dx,\\
&\qquad\qquad i = 1, 2, \dots, n.
\end{aligned} \tag{13}$$

Suppose that $u_n^m(x)$ is an approximation of function $u_n(x,t)$ at $t = t_m$, $2 \le m \le M$. Using formula (9) and the values u_{ni}^m, $i = 1, 2, \dots, n$, assume

$$u_n^m(x) = \sum_{i=1}^{n} u_{ni}^m \sin\frac{i\pi}{L}x. \tag{14}$$

3.3. Solution of the discrete system-iteration method

The last part of the algorithm is aimed at solving the discrete system of nonlinear equations (12), (13). It is assumed that the counting if performed levelwise, more precisely, knowing the results for the preceding two levels, on the m-th time level, $2 \le m \le M$, we have to solve the nonlinear system of algebraic equations (12), (13) with respect to the values u_{ni}^m, $i = 1, 2, \dots, n$. For this, we use the Jacobi type iteration method of the form

$$8\Big(\frac{L}{\tau i\pi}\Big)^2 u_{ni,k+1}^m + \Big[2\alpha + \frac{\beta L}{2}\Big(\frac{i\pi}{L}\Big)^2\big((u_{ni,k+1}^m)^2 + (u_{ni}^{m-1})^2\big)$$
$$+\frac{\beta L}{2}\sum_{\substack{j=1\\ j\neq i}}^{n}\Big(\frac{j\pi}{L}\Big)^2\big((u_{nj,k}^m)^2 + (u_{nj}^{m-1})^2\big)$$
$$+\frac{\beta}{L}\sum_{l=0}^{1}\big(-\mu(t_{m-l}) + \nu(t_{m-l})\big)^2\Big](u_{ni,k+1}^m + u_{ni}^{m-1}) = 8\Big(\frac{L}{\tau i\pi}\Big)^2 g_{ni}^m, \tag{15}$$
$$k = 0, 1, \dots, \quad i = 1, 2, \dots, n,$$

where $u^m_{ni,k}$ is the k-th iteration approximation of u^m_{ni}, $k=0,1,\dots$. In order not to complicate the discussion, here we assume that u^{m-2}_{ni} and u^{m-1}_{ni}, $i=1,2,\dots,n$, are defined with such accuracy that the corresponding errors can be neglected. Besides, let's assume $u^m_{ni,0}=u^{m-1}_{ni}$, $m=2,3,\dots,M$, $i=1,2,\dots,n$.

The equality (15) is a cubic equation with respect to the sought value $\frac{i\pi}{L}u^m_{ni,k+1}$ on $(k+1)$-th iteration step for each i. Therefore, to simplify the iteration process, let us take into consideration that according the Cardano formula [17], a priori real root for cubic equation

$$z^3+Az^2+Bz+C=0 \tag{16}$$

is equal to

$$z=-\frac{A}{3}+\left[-\frac{S}{2}+\left(\frac{S^2}{4}+\frac{R^3}{27}\right)^{\frac12}\right]^{\frac13}-\left[\frac{S}{2}+\left(\frac{S^2}{4}+\frac{R^3}{27}\right)^{\frac12}\right]^{\frac13}, \tag{17}$$

where

$$R=-\frac{A^2}{3}+B, \quad S=\frac{2A^3}{27}-\frac{AB}{3}+C. \tag{18}$$

Let us rewrite (15) in the form (16) as follows

$$\left(\frac{i\pi}{L}u^m_{ni,k+1}\right)^3+a^{m-1}_{ni}\left(\frac{i\pi}{L}u^m_{ni,k+1}\right)^2+b^m_{ni,k}\left(\frac{i\pi}{L}u^m_{ni,k+1}\right)+c^m_{ni,k}=0, \tag{19}$$
$$k=0,1,\dots, \quad i=1,2,\dots,n,$$

where

$$\begin{gathered}
a^{m-1}_{ni}=\frac{i\pi}{L}u^{m-1}_{ni}, \quad b^m_{ni,k}=d^m_{ni,k}+\left(\frac{i\pi}{L}u^{m-1}_{ni}\right)^2+\frac{16}{\beta L}\left(\frac{L}{\tau i\pi}\right)^2,\\
c^m_{ni,k}=\frac{i\pi}{L}u^{m-1}_{ni}\left(d^m_{ni,k}+\left(\frac{i\pi}{L}u^{m-1}_{ni}\right)^2\right)-\frac{16}{\beta L}\left(\frac{L}{\tau i\pi}\right)^2\frac{i\pi}{L}g^m_{ni},\\
d^m_{ni,k}=\frac{4\alpha}{\beta L}+\sum_{\substack{j=1\\ j\neq i}}^{n}\left(\left(\frac{j\pi}{L}u^m_{nj,k}\right)^2+\left(\frac{j\pi}{L}u^{m-1}_{nj}\right)^2\right)\\
+\frac{2}{L^2}\sum_{l=0}^{1}\big(-\mu(t_{m-l})+\nu(t_{m-l})\big)^2.
\end{gathered} \tag{20}$$

(19) is a special type of equation (16). Applying formula (17) and notations (18)–(20), we obtain the required following formula, in which

$\frac{i\pi}{L} u^m_{ni,k+1}$ is expressed through the known values

$$\frac{i\pi}{L} u^m_{ni,k+1} = -\frac{a^{m-1}_{ni}}{3} + \sum_{p=0}^{1} (-1)^p \left((-1)^{p+1} \frac{s^m_{ni,k}}{2} + \left(\frac{(s^m_{ni,k})^2}{4} + \frac{(r^m_{ni,k})^3}{27} \right)^{\frac{1}{2}} \right)^{\frac{1}{3}},$$
$$k = 0, 1, \ldots, \quad i = 1, 2, \ldots, n,$$

where

$$r^m_{ni,k} = -\frac{(a^{m-1}_{ni})^2}{3} + b^m_{ni,k}, \quad s^m_{ni,k} = \frac{2(a^{m-1}_{ni})^3}{27} - \frac{a^{m-1}_{ni} b^m_{ni,k}}{3} + c^m_{ni,k}.$$

For fixed $n \geq 1$, $2 \leq m \leq M$ and $k = 1, 2, \ldots,$ in formula (14) we replace u^m_{ni} with the k-th iteration approximation of this value, i.e., of $u^m_{ni,k}$, $i = 1, 2, \ldots, n$, and instead of $u^m_n(x)$ we write $u^m_{n,k}(x)$. So, let

$$u^m_{n,k}(x) = \sum_{i=1}^{n} u^m_{ni,k} \sin \frac{i\pi}{L} x. \tag{21}$$

Formula (21) represents the solution of problem (6), (7), which is obtained by the algorithm described here at k-th iteration step.

4 Appropriate Solution of Problem (1), (2)

Assume that problem (6), (7) is solved by the algorithm considered herein. Let us find the values of the solution of problem (1), (2) at time grid notes $t_m = m\tau$, $m = 0, 1, \ldots, M$.

For t_0 this value is exact, $w(x, t_0) = w^0(x)$, and for t_1 it is approximate,

$$w(x, t_1) \approx w^0(x) + \tau w^1(x) + \frac{r^2}{2} \left[f(x, 0) - \left(\alpha + \beta \int_0^L (w^{0\prime}(x))^2 dx \right) w^{0\prime\prime}(x) \right],$$

since it is obtained using the Taylor's formula.

As follows from equalities (3) and (21) for fixed $n \geq 1$, $2 \leq m \leq M$, and $k = 1, 2, \ldots,$ the corresponding approximation of the solution of problem (1), (2), which we denote by $w^m_{n,k}(x)$ has the form

$$w^m_{n,k}(x) = \sum_{i=1}^{n} u^m_{ni,k} \sin \frac{i\pi}{L} x + \left(\frac{L - x}{L} \mu(t_m) + \frac{x}{L} \nu(t_m) \right).$$

5 Conclusion

Reformulation of the problem justified itself. As a result, it became possible to get the convenient system of equations at the stage of space discretization. Also, no significant difficulties were created at the stage of time discretization and for application of the iteration method at the stage of solution the system of discrete equation.

Should also be noted useful to apply the Cardano formula to solve the system of discrete equations with cubic nonlinearity.

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