BOUNDARY VALUE PROBLEMS FOR A CLASS OF LINEAR HOMOGENEOUS FIRST-ORDER HYPERBOLIC SYSTEMS

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Abstract

Boundary value problems for a class of linear homogeneous first-order hyperbolic systems are considered. Necessary and sufficient conditions, imposed on the boundary coefficients that ensure the correctness of the problem are found. It is shown what type of violation of the correctness of the problem occurs when these conditions are not fulfilled. It is also shown what changes in the initial conditions should be made to make the problem correct. In the case of a correctly posed problem, the solution is written out explicitly.

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On the plane Oxt, in the semistrip $D: \ 0 < x < l,\ t > 0$, for the first-order linear system

$$\frac{\partial u}{\partial t} + A\,\frac{\partial u}{\partial x} + Bu = F(x,t), \ \ (x,t) \in D, \tag{1_0}$$

where A and B are given n-order quadratic matrices, $F = (F_1, \dots, F_n)$ is given, while $u = (u_1, \dots, u_n)$ is unknown vector-functions,consider a boundary value problem set as follows: in the domain D find a regular solution $u = (u_1, \dots, u_n)$ to the system (1_0) satisfying the following boundary conditions

$$(M_0 u)(x,o) = \varphi(x), \ \ 0 \le x \le l, \tag{2_0}$$

$$(M_1 u)(x,o) = \mu_1(x), \ \ t \ge 0, \tag{3_0}$$

$$(M_2 u)(x,o) = \mu_2(x), \ \ t \ge 0, \tag{4_0}$$

where M_i are given $m_i \times n$-dimensional matrices, $i = 0, 1, 2$; $\varphi = (\varphi_1, \dots, \varphi_{m_0})$ and $\mu_i = (\mu_{i1}, \dots, \mu_{m_i})$, $i = 1, 2$, are given vector-functions. Here the number m_i, $0 \le m_i \le n$, shows the measure with which the boundary conditions occupy the corresponding part of the boundary D. Particularly, $m_i = 0$ means that the corresponding part of the boundary D is completely free from boundary conditions.

Remark 1. In the boundary conditions (2_0), (3_0) and (4_0), the selection of matrices M_i, $i = 0, 1, 2$, for correctness of the boundary value problem (1_0)–(4_0) is essentially depended on the type of the system (1_0), i.e. whether the system (1_0) is elliptic, hyperbolic, parabolic or of compound type. In the case of hyperbolicity, the numbers m_i depend on the number of positive/negative characteristic roots of the matrix A and, moreover, the correctness of the boundary value problem (1_0)–(4_0) depends on the structure of matrices A and M_i. When the system (1_0) is hyperbolic, i.e. when all characteristic roots of matrix A are real, many authors have investigated the initial-boundary problem when in the condition (2_0) the matrix M_0 is unit. However, initial-boundary problem, as a rule, is being considered when the system (1_0) is splitted in the main part, i.e. the matrix A is diagonal and for certain conditions set on problem's data the existence and uniqueness of a solution are shown (see works [1, 2, 3, 4, 5, 6] and therein cited literature).

Below we consider the case when the system (1_0) is not splitted in the main part and it is strictly hyperbolic. For simplicity we consider the case $n = 2$, the matrix A is not diagonal and it has real characteristic roots of different signs, and $B = 0$ and $F = 0$. In this case the problem (1_0)–(4_0) can be rewritten as

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = 0, \quad (x,t) \in D, \quad 0 < x < l, \quad t > 0, \tag{1}$$

$$u(x,0) = \varphi(x), \quad 0 \le x \le l, \tag{2}$$

$$(\alpha_1 u_1 + \beta_1 u_2)(0,t) = \mu_1(t), \quad t \ge 0, \tag{3}$$

$$(\alpha_2 u_1 + \beta_2 u_2)(l,t) = \mu_2(t), \quad t \ge 0, \tag{4}$$

where (1) is a strictly hyperbolic system, particularly, let us assume that the second order quadratic matrix A with constant coefficients has two real characteristic roots $\lambda_1 > 0$ and $\lambda_2 < 0$, $u = u(x,t) = (u_1(x,t), u_2(x,t))$ is unknown vector-function from the class C^1; $\varphi = (\varphi_1, \varphi_2)$, $\mu = (\mu_1, \mu_2)$ are given vector-functions from the class C^1; α_i and β_i, $i = 1, 2$, are given constants and $|\alpha_i| + |\beta_i| \ne 0$, $i = 1, 2$. Besides, in the points $(0,0)$ and

$O'(l,0)$ there hold the following agreement conditions:

$$\begin{cases} \alpha_1\varphi_1(0)+\beta_1\varphi_2(0)=\mu_1(0), \quad \alpha_2\varphi_1(l)+\beta_2\varphi_2(l)=\mu_2(0), \\ \mu_1'(0)+(\alpha_1 a_{11}+\beta_1 a_{21})\varphi_1'(0)+(\alpha_1 a_{12}+\beta_1 a_{22})\varphi_2'(0)=0, \\ \mu_2'(0)+(\alpha_2 a_{11}+\beta_2 a_{21})\varphi_1'(l)+(\alpha_2 a_{12}+\beta_2 a_{22})\varphi_2'(l)=0. \end{cases} \tag{5}$$

Below, we give the necessary and sufficient conditions for coefficients in boundary conditions which ensure the correctness of the posed problem, besides, we show what kind of problem correctness violations occur when these conditions are not fulfilled. In this case, we also show what changes to the initial conditions are needed to make the problem correct. If the problem is set correctly, the solution can be written in explicit form.

Let $v_i \in \mathbb{R}^2 \setminus \{0\}$ be an eigen vector of the matrix A corresponding to the characteristic root λ_i, i.e. $Av_i = \lambda_i v_i$, $i=1,2$. For the second order quadratic matrix $K=(v_1,v_2)$ we have $K^{-1}AK = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and with respect to the new vector-function $w = K^{-1}u$ the problem (1)–(4) takes the form:

$$\frac{\partial W_i}{\partial t} + \lambda_i \frac{\partial W_i}{\partial x} = 0, \ \ (x,t) \in D, \ \ i=1,2, \tag{6}$$

$$w_i(x,0) = \psi_i(x), \ \ 0 \le x \le l, \ \ i=1,2, \tag{7}$$

$$\big[(\alpha_1 k_{11}+\beta_1 k_{21})w_1 + (\alpha_1 k_{12}+\beta_1 k_{22})w_2\big](0,t) = \mu_1(t), \ \ t \ge 0, \tag{8}$$

$$\big[(\alpha_2 k_{11}+\beta_2 k_{21})w_1 + (\alpha_2 k_{12}+\beta_2 k_{22})w_2\big](l,t) = \mu_2(t), \ \ t \ge 0, \tag{9}$$

where $K=(k_{ij})_{i,j=1}^2$, $\psi=(\psi_1,\psi_2)=K^{-1}\varphi$.

Theorem. *Let $\varphi \in C^1([0,l])$, $\mu \in C^1([0,\infty))$ and the agreement conditions* (5) *be fulfilled. Then for the correctness of the problem* (6)–(9) *and, therefore, the problem* (1)–(4)*, in the class C^1 it is necessary and sufficient the validity of the following conditions*

$$a = \alpha_1 k_{11} + \beta_1 k_{21} \neq 0, \quad d = \alpha_2 k_{12} + \beta_2 k_{22} \neq 0. \tag{10}$$

Remark 2. It should be noted that if at least one from the rest coefficients in the boundary conditions (8), (9)

$$b = \alpha_1 k_{12} + \beta_1 k_{22} \ \text{ and } \ c = \alpha_2 k_{11} + \beta_2 k_{21}$$

equals zero, then in the case when the condition (10) is fulfilled the unique solution of the problem (6)–(9) can be written by simple formula containing finite number of addends dependent on vector-functions φ and μ. When all four coefficients a, b, c and d are nonzero, then the structure of solution of the problem (6)–(9) is much complex. In this case the finite number of addends in the formula of the solution $u = u(x,t)$ depends on the point (x,t) and converges to infinity for $t \to \infty$.

As mentioned above, in the case of violation of the conditions (10) the problem (6)–(9) and, therefore, the initial problem (1)–(4), is not set correctly. For example, consider the case when $a = 0$ and $c \neq 0$. Note that since $|\alpha_i| + |\beta_i| \neq 0$, $i = 1, 2$, and $\det K \neq 0$, then $|a| + |b| \neq 0$, $|c| + |d| \neq 0$ and, therefore, $|b| \neq 0$. In this case, the initial conditions (2) in the problem (1)–(4) are completely unnecessary, and for making the problem correct the initial conditions (2) must be removed from the statement of the problem. Then the boundary problem (1), (3), (4), i.e., the problem

$$\begin{gathered}
\frac{\partial u}{\partial t} + A\,\frac{\partial u}{\partial x} = 0, \;\; (x,t) \in D, \;\; 0 < x < l, \;\; t > 0, \\
(\alpha_1 u_1 + \beta_1 u_2)(0,t) = \mu_1(t), \;\; t \geq 0, \\
(\alpha_2 u_1 + \beta_2 u_2)(l,t) = \mu_2(t), \;\; t \geq 0,
\end{gathered}$$

will be set correctly. In these conditions, i.e. when $a = 0$ and $c \neq 0$, the following boundary problem will be also correctly set

$$\begin{gathered}
\frac{\partial u}{\partial t} + A\,\frac{\partial u}{\partial x} = 0, \;\; (x,t) \in D, \;\; 0 < x < l, \;\; t > 0, \\
(\gamma_1 u_1 + \gamma_2 u_2)(x,0) = \varphi_1(x), \;\; 0 \leq x \leq l, \\
(\alpha_1 u_1 + \beta_1 u_2)(0,t) = \mu_1(t), \;\; t \geq 0, \\
(\alpha_2 u_1 + \beta_2 u_2)(l,t) = \mu_2(t), \;\; t \geq \frac{l}{\lambda_1}\,,
\end{gathered}$$

where $\gamma_1 k_{11} + \gamma_2 k_{21} \neq 0$ and the right hand lateral side $\{x = l, \; t \geq 0\}$ of the semistrip $D: \; 0 < x < l$, $t > 0$, is loaded with boundary condition only partially, when $t \geq \frac{l}{\lambda_1}$.

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