ON ONE PROBLEM IN THE PLANE THEORY OF VISCOELASTICITY FOR POLYGONAL AREA WITH A CIRCULAR HOLE

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Abstract

The problem of the plane theory of viscoelasticity for a convex polygon with a circular hole is considered according to the Kelvin-Voigt model. It is assumed that absolutely rigid smooth stamps are applied on the sides of the polygon on which normal compressive forces are applied with given main vectors (or constant normal displacements are given) and the internal boundary is subject to a normal compressive force (pressure) of a given intensity.

The purpose of this work is to determine the elastic equilibrium of a viscoelastic plate occupying a given region using the Kelvin-Voigt model. To solve the problem, methods of conformal mapping and boundary problems of analytical functions are used. The required complex potentials are constructed efficiently (in analytical form).

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1 Statement of the Problem

Let a viscoelastic plate on the plane of a complex variable occupy a doubly connected domain S, bounded by a convex polygon (A) and a circle of unit radius L_0 (i.e., we have a polygonal domain with a circular hole). Let us denote by L_1 the boundary of the polygon (A), i.e.

$$L_1 = \bigcup_{k=1}^{n} L_k^{(1)}, \quad L_k^{(1)} = A_k A_{k+1} \quad (k = \overline{1,n}, \ \ A_{n+1} = A_1),$$

The values of the internal angles of the domain S at the vertices $A_k,\ k = \overline{1,n}$ will be denoted by $\pi\alpha_k^0$. The angle between the axis x and the external normal to the contour L_1 at a point will σ be denoted by $\alpha(\sigma)$, i.e. $\alpha(\sigma) = \alpha_0^{(k)} = const,\ \ \sigma \in L_1^{(k)}$.

Let us assume that straight absolutely rigid smooth stamps with known main normal forces $N_k\ \ (k = \overline{1,n})$ are applied to the sides $L_k^{(1)}$ and the internal boundary is subject to uniformly distributed normal pressure P_0 (This type of external load somewhat simplifies some calculations and does not change the essence of the solution to the problem).

The purpose of this work is to determine complex potentials characterizing the distribution of stresses and displacements in the plate according to the Kelvin-Voigt model.

Similar problems of the plane theory of elasticity for finite doubly connected polygonal regions are considered in [1-3].

2 Solution of the problemm

The problem is solved by the methods of conformal mappings and the theory of boundary value problems of analytic functions. Based on the Kolosov-Muskhelishvili formulas [6], the problem of finding the required complex potentials is reduced to the Riemann-Hilbert boundary value problem for a circular ring.

Due to the fact that the given domain is doubly connected, it is advisable to use functions $\Phi(z,t)$ and $\Psi(z,t)$, which are also Unique in the case of a multiply connected domain.

Let us present some results arising from the works [4-6].

The function $z = \omega(\zeta)$ conformally maps the circular ring $D = \{1 < |\zeta| < R\}$ onto the domain S. Its derivative is the solution of the Riemann-Hilbert problem for the circular ring D

$$Re\left[i\omega'(\xi)\right] = 0,\ \xi \in l_0; \quad Re\left[i\xi e^{-i\alpha(\xi)}\omega'(\xi)\right] = 0,\ \xi \in l_1,$$

where l_0 and l_1 are the inverse images of contours L_0 and L_1, respectively, i.e.

$$l_0 = \{|\zeta| = 1\}\,; \quad l_1 = \{|\zeta| = R\}\,,$$

and subject to

$$\prod_{k=1}^{n}\left(\frac{a_k}{R}\right)^{\alpha_k^0 - 1} = 1,$$

have the form

$$\omega'(\zeta)=K^0\prod_{k=1}^{n}\left(1-\frac{a_k}{\zeta}\right)^{\alpha_k^0-1}\prod_{j=1}^{\infty}\prod_{k=1}^{n}\left(1-\frac{\zeta}{R^{2j}a_k}\right)^{\alpha_k^0-1} \times\left(1-\frac{a_k}{R^{2j}\zeta}\right)^{\alpha_k^0-1}, \tag{1}$$

where K^0 real constant $a_k=\omega^{-1}(A_k),\ \ k=\overline{1,n}$.

The first and second basic boundary value problems of the viscoelasticity plane S for the Kelvin-Voigt linear model have the following forms

$$\varphi(\sigma,t)+\sigma\overline{\varphi'(\sigma,t)}+\overline{\psi(\sigma,t)}=i\int_{\sigma_0}^{\sigma}(X_n+iY_n)\,ds+C_1+iC_2, \tag{2}$$

$$\Gamma\varphi(\sigma,t)-\mathrm{M}\left[\varphi(\sigma,t)+\sigma\overline{\varphi'(\sigma,t)}+\overline{\psi(\sigma,t}\right]=2\mu^*(u+iv), \tag{3}$$

$$\sigma\in L_0\bigcup L_1$$

here and then the coordinate t is the parameter of the time, Γ and M are operators of the time t

$$\begin{aligned}&\Gamma\varphi(\sigma,t)=\textstyle\int_0^t\left[\text{æ}^*e^{k(\tau-t)}+2e^{m(\tau-t)}\right]\varphi(\sigma,\tau)d\tau,\\ &\mathrm{M}\left[\varphi(\sigma,t)+\sigma\overline{\varphi'(\sigma,t)}+\overline{\psi(\sigma,t)}\right]\\ &=\textstyle\int_0^t e^{m(\tau-t)}\left[\varphi(\sigma,\tau)+\sigma\overline{\varphi'(\sigma,\tau)}+\overline{\psi(\sigma,\tau)}\right]d\tau,\quad \sigma\in L.\end{aligned} \tag{4}$$

Considering the equality

$$X_n+iY_n=(N+iT)\,e^{i\alpha(\sigma)}=-i(N+iT)\frac{dt}{ds},$$

from (2) by differentiation with respect σ to we obtain

$$\begin{aligned}&\tfrac{d}{d\sigma}\left[\varphi(\sigma,t)+\sigma\overline{\varphi'(\sigma,t)}+\overline{\psi(\sigma,t)}\right]=\Phi(\sigma,t)+\overline{\Phi(\sigma,t)}+\\ &\quad+\overline{\sigma}_s{}'^2\left[\sigma\overline{\Phi'(\sigma,t)}+\overline{\psi(\sigma,t)}\right]=N+iT,\end{aligned} \tag{5}$$

where $\Phi(\sigma,t)=\varphi'(\sigma,t),\ \ \Psi(\sigma,t)=\psi'(\sigma,t)$.

Considering (5) and taking into account the equalities

$$\begin{aligned}&u+iv=(v_n+iv_\tau)e^{i\alpha(\sigma)},\ \ v_n=v_n^{(j)}=const,\ \ \sigma\in L_1^{(j)},\ \ (j=\overline{1,n}),\\ &v_n=V_n^{(0)}\ \ =const,\ \sigma\in L_0;\ \ e^{i\alpha(\sigma)}=\sigma,\ \sigma\in L_0;\\ &v_\tau=0,\ T(\sigma)=0,\ \sigma\in L_0\bigcup L_1,\end{aligned}$$

from (3) by differentiation with respect σ to we obtain

$$\Gamma\Phi(\sigma,t)-\mathrm{M}\left[N+iT\right]=\begin{cases}2\mu^* v_n^{(0)}, & \sigma\in L_0,\\ 0, & \sigma\in L_1.\end{cases} \tag{6}$$

If we take into account that $N=P_0,\ \sigma\in L_0$ and $T=0,\ \sigma\in L_1$, from (6) we have

$$Re\Gamma\Phi(\sigma,t)=P(t),\ \sigma\in L_0;\ \ Im\Gamma\Phi(\sigma,t)=0,\ \sigma\in L_1, \tag{7}$$

where

$$P(t)=P_0F(t)+2\mu^* v_n^{(0)};\quad F(t)=\frac{1}{m}\left[1-e^{-mt}\right]. \tag{8}$$

Mapping domain S onto a circular ring $D=\{1<|\zeta|<R\}$ (see point 1), with respect to the function

$$\Omega(\zeta,t)=\Gamma\Phi_0(\zeta,t),\quad \Phi_0(\zeta,t)=\Phi\left[\omega(\zeta),t\right] \tag{9}$$

from (7) we obtain the Riemann-Hilbert boundary value problem for a circular ring D

$$Re\left[\Omega(\eta,t)-P(t)\right]=0,\eta\in l_0;\quad Im\left[\Omega(\eta,t)-P(t)\right]=0,\ \eta\in l_1, \tag{10}$$

where l_0 and l_1 are inverse images of boundaries L_0 and L_1.

Since problem (10) has only a trivial solution, we will have $\Omega(\zeta,t)=P(t),\ \zeta\in D$ and, therefore, to determine function $\Phi_0(\zeta,t)$ based on (4) and (9) we obtain the integral equation

$$\int_0^t\left[æ^* e^{k(\tau-t)}+2e^{m(\tau-t)}\right]\Phi_0(\zeta,\tau)d\tau=P(t). \tag{11}$$

By differentiating (11) with respect to t and then adding the resulting equality with (11) multiplied by m, we will have

$$(m-k)æ^*\int_0^t e^{k\tau}\Phi_0(\zeta,\tau)d\tau+(æ^*+2)e^{kt}\Phi_0(\zeta,t)=(P_0+mv_n^{(0)})e^{kt}. \tag{12}$$

From (12) by differentiation with respect to t we obtain a differential equation of the first kind

$$\dot{\Phi}_0(\zeta,t)+a\Phi_0(\zeta,t)=b, \tag{13}$$

where

$$a=\frac{\mathrm{m}æ^*+2k}{æ^*+2},\qquad b=\frac{k(P_0+mv_n^{(0)})}{æ^*+2} \tag{14}$$

(the dot over $\Phi_0(\zeta, t)$ means the derivative with respect to t).

From (12) we have

$$\Phi_0(\sigma, 0) = \frac{b}{k}. \tag{15}$$

The solution to equation (13), under the initial condition (15) has the form

$$\Phi_0(\zeta, t) = b\left[\frac{1}{a} + \left(\frac{1}{k} - \frac{1}{a}\right)e^{-at}\right], \tag{16}$$

where a and b are defined by formula (14).

After function $\Phi_0(\zeta, t)$ is found, to define the function

$$\Psi_0(\zeta, t) = \Psi\left[\omega(\zeta), t\right]$$

Let us use the boundary condition (6), which after the conformal mapping, taking into account (5), will be written in the form

$$\Gamma\overline{\Phi_0(\eta, t)} - \mathrm{M}\left\{\Phi_0(\eta, t) + \overline{\Phi_0(\eta, t)} - \tfrac{\eta^2}{\omega'(\eta)}\left[\overline{\omega(\eta)}\Phi_0'(\eta, t) + \omega'(\eta)\Psi_0(\eta, t)\right]\right\}$$
$$= \begin{cases} 2\mu^* v_n^{(0)}, & \sigma \in l_0, \\ 0, & \sigma \in l_1, \end{cases} \quad \eta \in l = l_0 \bigcup l_1. \tag{17}$$

Based on (16), from (17) we obtain

$$Im\Omega_0(\eta, t) = 0, \quad \eta \in l_0 \bigcup l_1, \tag{18}$$

where

$$\Omega_0(\zeta, t) = \mathrm{M}\left[\zeta^2\omega'^2(\zeta)\Psi_0(\zeta, t)\right]. \tag{19}$$

The solution to problem (18) has the form $\Omega_0(\zeta, t) = K_1$ where K_1 is a real constant.

Thus, for Definition $\Psi_0(\zeta, t)$ we obtain the equation

$$\mathrm{M}\left[\zeta^2\omega'^2(\zeta)\Psi_0(\zeta, t)\right] = K_1. \tag{20}$$

Taking into account the form of operator M, from (20) we easily obtain

$$\Psi_0(\zeta, t) = \frac{mK_1}{\zeta^2\omega'^2(\zeta)}.$$

From (14) and (16) we conclude that expression $\Phi_0(\zeta, t)$ involves both P_0 and $v_n^{(0)}$, and the condition of the problem implies that L_0 is set to one of them. Therefore, it is in our interests to find the relationship between these quantities. Let us assume that at the initial moment the value L_0 is set to P_0. Then we will have

$$X_x = Y_y = P_0, \quad X_x + Y_y = 4Re\Phi(\sigma, t), \quad Re\Phi(\sigma, t) = \frac{P_0}{2}, \quad \sigma \in L_0$$

and condition (6), taking into account (4), will be written in the form

$$\frac{P_0 æ^*}{2} \int_0^t e^{k(\tau - t)} d\tau = 2\mu^* v_n^{(0)},$$

from which we easily obtain

$$v_n^{(0)} = \frac{æ^* P_0}{4\mu^* k}.$$

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