ANALYTICAL SOLUTION OF BVP FOR AREA BOUNDED BY PARABOLA WITH NORMAL LOAD

N. Zirakashvili

I. Vekua Institute of Applied Mathematics of
Iv. Javakhishvili Tbilisi State University
0186 University Street 2, Tbilisi, Georgia
natela.zirakashvili@gmail.com

(Received 23.01.2022; accepted 06.04.2022)

Abstract

In this paper internal boundary value problem of elastic equilibrium of the homogeneous isotropic body bounded by coordinate lines of the parabolic coordinate system is considered, when on the parabolic border normal stress is given. The exact solution is obtained by the method of separation of variables. Using the MATLAB software, the numerical results are obtained at some characteristic points of the body and relevant 2D and 3D graphs are presented.

Keywords and phrases: Internal boundary value problem, Separation of variables, Parabolic coordinates.

AMS subject classification (2010): 74B05, 74G10.

1 Introduction

To solve the boundary value problems and boundary contact problems in areas with a curved boundary, it is more purposeful to consider such problems in the corresponding curvilinear coordinate system. For example, for a circle and its parts the tasks are considered in the polar coordinate system [1-5], for the ellipse and its parts, and area bounded by hyperbola the tasks are considered in the elliptical coordinate system [6-15], for an areas with a circle with different centers and radiuses the tasks are considered in the bipolar coordinate system [16-18], and the problems for the areas with parabolic boundaries are considered in the parabolic coordinates [15], [19-20]. The above-mentioned tasks are solved by both analytical and numerical methods.

In the current paper the boundary value problem is considered in the parabolic coordinate system ξ, η; $-\infty < \xi < \infty$, $0 \leq \eta < \infty$ (see appendix A). In the parabolic coordinates the equilibrium equations system and Hooke's law are written. Analytical (exact) solution of 2D elasticity problems is built in a region bounded by the coordinate lines of the parabolic coordinate system. Here is presented internal boundary value problem of elastic equilibrium of a homogeneous isotropic body bounded by the coordinate lines of the parabolic coordinate system, when a normal stress is given at the parabolic boundary. Exact solution is obtained using the method of separation of variables. Numerical results and corresponding graphs of above-mentioned problem are presented.

2 Equilibrium equations and Hooke's law in parabolic coordinates

In parabolic coordinates ξ,η, equilibrium equations with respect to the function $D,\ K,\ \bar{u},\ \bar{v}$ and Hooke's law can be written as [21]

$$\begin{array}{ll} \text{a)}\ D_{,\xi}-K_{,\eta}=0, & \text{c)}\ \bar{u}_{,\xi}+\bar{v}_{,\eta}=\frac{\kappa-2}{\kappa\mu}h_0^2D, \\ \text{b)}\ D_{,\eta}+K_{,\xi}=0, & \text{d)}\ \bar{v}_{,\xi}-\bar{u}_{,\eta}=\frac{1}{\mu}h_0^2K, \end{array} \tag{1}$$

$$\begin{array}{l} \sigma_{\eta\eta}=h_0^{-1}\left[\lambda\bar{u}_{,\xi}+(\lambda+2\mu)\,\bar{v}_{,\eta}+\left[(\lambda+\mu)-\mu h_0^{-2}\right](\xi\bar{u}+\eta\bar{v})\right], \\ \sigma_{\xi\xi}=h_0^{-1}\left[(\lambda+2\mu)\,\bar{u}_{,\xi}+\lambda\bar{v}_{,\eta}+\left[(\lambda+\mu)+\mu h_0^{-2}\right](\xi\bar{u}+\eta\bar{v})\right], \\ \tau_{\xi\eta}=\mu h_0^{-1}\left[(v_{,\xi}+u_{,\eta})-h_0^{-2}\,(\xi\bar{v}+\eta\bar{u})\right], \end{array} \tag{2}$$

where $\bar{u}=\frac{hu}{c^2},\ \bar{v}=\frac{hv}{c^2};\ h_0=\sqrt{\xi^2+\eta^2}, h=h_\xi=h_\eta=c\sqrt{\xi^2+\eta^2}$ are Lamé coefficients, $u,\ v$ are components of the displacement vector at tangents to the coordinate lines $\eta,\ \xi$; $\frac{\kappa-2}{\kappa\mu}D$ is the divergence of the displacement vector, $\frac{K}{\mu}$ is the rotor of the displacement vector; $\sigma_{\xi\xi},\ \sigma_{\eta\eta}$ and $\tau_{\xi\eta}=\tau_{\eta\xi}$ are normal and tangential stresses; subscripts ξ,η denote partial derivatives with respect to the corresponding coordinates; $\lambda=\frac{E\nu}{(1+\nu)(1-2\nu)}$, $\mu=\frac{E}{2(1-\nu)}$ are elastic Lamé constants; $\kappa=4\,(1-\nu)$; ν is the Poisson's ratio and E is the modulus of elasticity.

3 Statement and solution of problem

3.1 Boundary conditions

Now let us formulate the following boundary value problem: in the domain $\Omega_1=\{0<\xi<\xi_1,\ \ 0<\eta<\eta_1\}$(see Fig.1) find a solution of the system of equilibrium equations (1) with respect to the unknowns $D,\ K,\ u,\ v$ using the boundary conditions:

$$\text{for}\ \ \eta=\eta_1:\quad \frac{h_0^2}{2\mu}\sigma_{\eta\eta}=P,\ \ \frac{h_0^2}{2\mu}\tau_{\xi\eta}=0. \tag{3}$$

$$\text{for}\ \ \eta=0:\quad \bar{u}=0,\quad \bar{v}_{,\eta}=0. \tag{4}$$

$$\text{for } \xi = 0: \quad \bar{v} = 0, \quad \bar{u}_{,\xi} = 0. \tag{5}$$

$$\text{for } \xi = \xi_1: \quad u = 0, \quad v = 0, \tag{6}$$

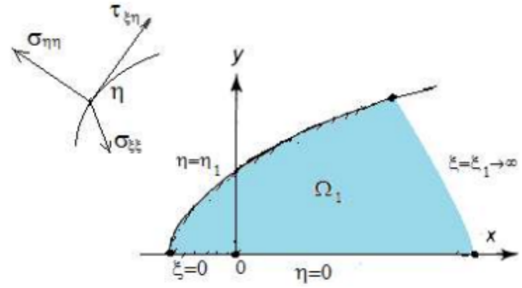


Figure 1: Area $\Omega_1 = \{0 < \xi < \xi_1,\ \ 0 < \eta < \eta_1\}$ bounded by the parabola and the line y=0.

Boundary conditions on the linear parts $\xi = 0$ and $\eta = 0$ of consideration area enables us to continue the solutions continuously in the domain, that is the mirror reflection of the consideration area in a relationship $y = 0$ line (see Fig.2).

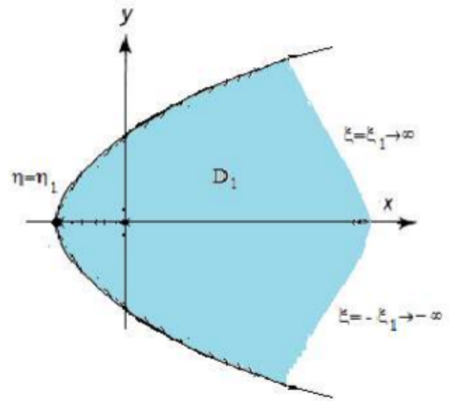


Figure 2: Area $D_1 = \{-\xi_1 < \xi < \xi_1,\ \ 0 < \eta < \eta_1\}$ bounded by parabola.

3.2 Solution of system of partial differential equations

We solve the system of partial differential equations (1).

We have introduce φ_1 harmonic function and if we take

$$\begin{aligned} &\text{a)} \quad D = \tfrac{\kappa\mu}{h_0^2}\left(\varphi_{1,\eta}\eta - \varphi_{1,\xi}\xi\right), \\ &\text{b)} \quad K = \tfrac{\kappa\mu}{h_0^2}\left(\varphi_{1,\eta}\xi + \varphi_{1,\xi}\eta\right), \end{aligned} \tag{7}$$

then (1a) and (1b) equations will be satisfied identically, while (1c) and (1d) equations will receive the following form:

$$\begin{aligned} &\text{a)} \quad \bar{u}_{,\eta} + \bar{v}_{,\xi} = (\kappa - 2)\left(\varphi_{1,\eta}\eta - \varphi_{1,\xi}\xi\right), \\ &\text{b)} \quad v_{,\xi} - \bar{u}_{,\eta} = \kappa\left(\varphi_{1,\eta}\xi + \varphi_{1,\xi}\eta\right), \end{aligned} \tag{8}$$

$$\begin{aligned} &\text{a)} \quad \bar{\text{u}}_{,\xi} + \bar{v}_{,\eta} = (\kappa - 2)\left(\varphi_{1,\eta}\eta - \varphi_{1,\xi}\xi\right), \\ &\text{b)} \quad \left(v - \kappa\varphi_1\eta\right)_{,\xi} = \left(\bar{u} + \kappa\varphi_1\xi\right)_{,\eta}, \end{aligned} \tag{9}$$

(9b) implies that theare exists a harmonic function ϕ of such type for which fulfil the following

$$\bar{u} = \phi_{,\xi} - \kappa\varphi_1\xi, \qquad \bar{v} = \phi_{,\eta} + \kappa\varphi_1\eta. \tag{10}$$

Considering (10), from equation (9a) we will obtain

$$\begin{aligned} h^2\Delta\phi = \phi_{,\xi\xi} + \phi_{,\eta\eta} = \kappa\varphi_1 + \kappa\varphi_{1,\xi}\xi - \kappa\varphi_1 - \kappa\varphi_{1,\eta}\eta \\ + (\kappa - 2)\left(\varphi_{1,\eta}\eta - \varphi_{1,\xi}\xi\right) = 2\left(\varphi_{1,\xi}\xi - \varphi_{1,\eta}\eta\right). \end{aligned} \tag{11}$$

General solution of the system (8) can be written in the form $\bar{u} = \psi_1$, $\bar{v} = \psi_2$, where

$$\psi_{1,\xi} + \psi_{2,\eta} = 0, \qquad \psi_{2,\xi} - \psi_{1,\eta} = 0.$$

The full solution of the equation system (8) is written in the following form

$$\bar{u} = \phi_{,\xi} - \kappa\varphi_1\xi + \psi_1, \qquad \bar{v} = \phi_{,\eta} + \kappa\varphi_1\eta + \psi_2, \tag{12}$$

where ϕ is the partial solution of (11).

If we take $\kappa = const$, then

$$\phi = \frac{\xi^2 - \eta^2}{2}\varphi_1$$

and (12) formula will receive the following form:

$$\bar{u} = \frac{\xi^2 - \eta^2}{2}\varphi_{1,\xi} - (\kappa - 1)\varphi_1\xi + \psi_1, \qquad \bar{v} = \frac{\xi^2 - \eta^2}{2}\varphi_{1,\eta} + (\kappa - 1)\varphi_1\eta + \psi_2.$$

From here

$$\begin{aligned} \bar{u} &= \left(\tfrac{\xi^2 - \eta^2}{2}\varphi_{1,\xi} + \xi\eta\varphi_{1,\eta}\right) - \xi\eta\varphi_{1,\eta} - (\kappa - 1)\varphi_1\xi + \psi_1, \\ \bar{v} &= \left(\tfrac{\xi^2 - \eta^2}{2}\varphi_{1,\eta} - \xi\eta\varphi_{1,\xi}\right) + \xi\eta\varphi_{1,\xi} + (\kappa - 1)\varphi_1\eta + \psi_2. \end{aligned}$$

Without loss of generality the expression in brackets can be taken to be zero, because we already have in $\bar{u}$ and $\bar{v}$ of the solutions Laplacian (we mean ψ_1 and ψ_2). Therefore, the solution of system (1) are given in the following form:

$$\begin{array}{l} \text{a)}\ \ h_0^2 D = \kappa\mu\left(\varphi_{1,\eta}\eta - \varphi_{1,\psi}\xi\right), \\ \text{b)}\ \ h_0^2 K = \kappa\mu\left(\varphi_{1,\eta}\xi + \varphi_{1,\xi}\eta\right), \\ \text{c)}\ \ \bar{u} = -\xi\eta\varphi_{1,\eta} - (\kappa-1)\,\varphi_1\xi + \psi_1, \\ \text{d)}\ \ \bar{v} = \xi\eta\varphi_{1,\xi} + (\kappa-1)\,\varphi_1\eta + \psi_2. \end{array} \tag{13}$$

Now we have to write down three versions of ψ_1 and ψ_2 function representation. In the first version

$$\begin{array}{l} \psi_1 = \bar{\varphi}_{1,\eta} + \tilde{\varphi}_{1,\eta} + \varphi_{2,\eta}, \\ \psi_2 = \bar{\varphi}_{1,\xi} + \tilde{\varphi}_{1,\xi} + \varphi_{2,\xi}, \end{array} \tag{14}$$

$\bar{\varphi}_1, \tilde{\varphi}_1, \varphi_2$ are harmonic functions, in addition, $\bar{\varphi}_1, \tilde{\varphi}_1$ are selected so that at $\eta = \alpha$, where $\alpha = \eta_1$ or $\alpha = \eta_2$, satisfy the following equations

$$\begin{array}{l} -\xi\eta\varphi_{1,\eta} - (\kappa-1)\,\varphi_1\xi + \bar{\varphi}_{1,\eta} + \tilde{\varphi}_{1,\eta} = 0, \\ \xi\eta\varphi_{1,\xi} + (\kappa-1)\,\varphi_1\eta + \bar{\varphi}_{1,\xi} + \tilde{\varphi}_{1,\xi} = 0. \end{array}$$

In the second version

$$\begin{array}{l} \psi_1 = -\alpha\left(\frac{\xi^2-(\eta-\alpha)^2}{2}\varphi_{1,\xi} + \xi\eta\varphi_{1,\eta}\right) + \frac{\xi^2-\eta^2}{2}\varphi_{2,\xi} + \xi\eta\varphi_{2,\eta}, \\ \psi_2 = \alpha\left(\xi\eta\varphi_{1,\xi} - \frac{\xi^2-(\eta-\alpha)^2}{2}\varphi_{1,\eta}\right) + \frac{\xi^2-\eta^2}{2}\varphi_{2,\eta} - \xi\eta\varphi_{2,\xi}\ , \end{array} \tag{15}$$

where φ_2 is the harmonic function.

In the third version

$$\begin{array}{l} \psi_1 = -\alpha^2\left(\frac{\xi^2-\eta^2}{2}\varphi_{1,\xi} + \xi\eta\varphi_{1,\eta}\right) + \frac{\xi^2-\eta^2}{2}\varphi_{2,\xi} + \xi\eta\varphi_{2,\eta}, \\ \psi_2 = \alpha^2\left(\xi\eta\varphi_{1,\xi} - \frac{\xi^2-\eta^2}{2}\varphi_{1,\eta}\right) + \frac{\xi^2-\eta^2}{2}\varphi_{2,\eta} - \xi\eta\varphi_{2,\eta}. \end{array} \tag{16}$$

Inserting (14) in (13c,d), we will get

$$\begin{array}{l} \text{a)}\ \ \bar{u} = -\xi\eta\varphi_{1,\eta} - (\kappa-1)\,\varphi_1\xi + \bar{\varphi}_{1,\eta} + \tilde{\varphi}_{1,\eta} + \varphi_{2,\eta}, \\ \text{b)}\ \ \bar{v} = \xi\eta\varphi_{1,\xi} + (\kappa-1)\,\varphi_1\eta + \bar{\varphi}_{1,\xi} + \tilde{\varphi}_{1,\xi} + \varphi_{2,\xi}. \end{array} \tag{17}$$

Inserting (15) in (13c,d), we will have

$$\begin{array}{l} \text{a)}\ \ \bar{u} = -\alpha\left(\frac{\xi^2-(\eta-\alpha)^2}{2}\varphi_{1,\xi} + \xi\eta\varphi_{1,\eta}\right) - \xi\eta\varphi_{1,\eta} - (\kappa-1)\,\varphi_1\xi \\ \qquad + \frac{\xi^2-\eta^2}{2}\varphi_{2,\xi} + \xi\eta\varphi_{2,\eta}, \\ \text{b)}\ \ \bar{v} = \alpha\left(\xi\eta\varphi_{1,\xi} - \frac{\xi^2-(\eta-\alpha)^2}{2}\varphi_{1,\eta}\right) + \xi\eta\varphi_{1,\xi} + (\kappa-1)\,\varphi_1\eta \\ \qquad + \frac{\xi^2-\eta^2}{2}\varphi_{2,\eta} - \xi\eta\varphi_{2,\xi}. \end{array} \tag{18}$$

Inserting (16) in (13c,d), we will get

$$\begin{aligned} &\text{a)}\ \bar{u} = -\alpha^2\left(\tfrac{\xi^2-\eta^2}{2}\varphi_{1,\xi}+\xi\eta\varphi_{1,\eta}\right)-\xi\eta\varphi_{1,\eta}-(\kappa-1)\,\varphi_1\xi\\ &\quad+\tfrac{\xi^2-\eta^2}{2}\varphi_{2,\xi}+\xi\eta\varphi_{2,\eta},\\ &\text{b)}\ \bar{v} = \alpha^2\left(\xi\eta\varphi_{1,\xi}-\tfrac{\xi^2-\eta^2}{2}\varphi_{1,\eta}\right)+\xi\eta\varphi_{1,\xi}+(\kappa-1)\,\varphi_1\eta\\ &\quad+\tfrac{\xi^2-\eta^2}{2}\varphi_{2,\eta}-\xi\eta\varphi_{2,\xi}. \end{aligned} \tag{19}$$

The solution is constructed using its general representation by two harmonic functions φ_1, φ_2. From formulas (17)-(19), after inserting $\alpha = \eta_1$ and making simple transformations, we will obtain:

$$\begin{aligned} \bar{u} &= -\left[\eta\left(\varphi_{1,\eta}-\varphi_{2,\xi}\right)+(\kappa-1)\,\varphi_1\right]\xi+\left[\tfrac{\eta_1^2}{\eta}\left(\varphi_{1,\xi}+\varphi_{2,\eta}\right)\right.\\ &\quad\left.-(\kappa-1)\,\varphi_2\right]\eta,\\ \bar{v} &= \left[\tfrac{\eta_1^2}{\eta}\left(\varphi_{1,\eta}-\varphi_{2,\xi}\right)+(\kappa-1)\,\varphi_1\right]\eta+\left[\eta\left(\varphi_{1,\xi}+\varphi_{2,\eta}\right)\right.\\ &\quad\left.-(\kappa-1)\,\varphi_2\right]\xi; \end{aligned} \tag{20}$$

$$\begin{aligned} D &= \tfrac{\kappa\mu}{h_0^2}\left[\left(\varphi_{1,\eta}-\varphi_{2,\xi}\right)\eta-\left(\varphi_{1,\xi}+\varphi_{2,\eta}\right)\xi\right],\\ K &= \tfrac{\kappa\mu}{h_0^2}\left[\left(\varphi_{1,\eta}-\varphi_{2,\xi}\right)\xi+\left(\varphi_{1,\xi}+\varphi_{2,\eta}\right)\eta\right], \end{aligned}$$

where $\frac{1}{h^2}\left(\varphi_{i,\xi\xi}+\varphi_{i,\eta\eta}\right)=0, \quad i=1,2.$

The stress tensor components can be written as:

$$\begin{aligned} \tfrac{h_0^2}{2\mu}\sigma_{\eta\eta} &= -\left[\tfrac{\eta_1^2}{\eta}\left(\varphi_{1,\xi\xi}+\varphi_{2,\xi\eta}\right)-\tfrac{\kappa}{2}\varphi_{1,\eta}-\tfrac{\kappa-2}{2}\varphi_{2,\xi}\right]\eta\\ &\quad+\left[\eta\left(\varphi_{1,\xi\eta}-\varphi_{2,\eta\eta}\right)+\tfrac{\kappa-2}{2}\varphi_{1,\xi}-\tfrac{\kappa}{2}\varphi_{2,\eta}\right]\xi\\ &\quad-\tfrac{\eta_1^2-\eta}{\xi^2+\eta^2}\left[\left(\varphi_{1,\eta}-\varphi_{2,\xi}\right)\eta-\left(\varphi_{1,\xi}+\varphi_{2,\eta}\right)\xi\right],\\ \tfrac{h_0^2}{2\mu}\tau_{\xi\eta} &= \left[\tfrac{\eta_1^2}{\eta}\left(\varphi_{1,\xi\eta}-\varphi_{2,\xi\xi}\right)+\tfrac{\kappa-2}{2}\varphi_{1,\xi}-\tfrac{\kappa}{2}\varphi_{2,\eta}\right]\eta\\ &\quad+\left[\eta\left(\varphi_{1,\xi\xi}+\varphi_{2,\xi\eta}\right)-\tfrac{\kappa}{2}\varphi_{1,\eta}-\tfrac{\kappa-2}{2}\varphi_{2,\xi}\right]\xi\\ &\quad-\tfrac{\eta_1^2-\eta}{\xi^2+\eta^2}\left[\left(\varphi_{1,\eta}-\varphi_{2,\xi}\right)\xi+\left(\varphi_{1,\xi}+\varphi_{2,\eta}\right)\eta\right],\\ \tfrac{h_0^2}{2\mu}\sigma_{\xi\xi} &= \left[\tfrac{\eta_1^2}{\eta}\left(\varphi_{1,\xi\xi}+\varphi_{2,\xi\eta}\right)-\tfrac{\kappa-4}{2}\varphi_{1,\eta}-\tfrac{\kappa+2}{2}\varphi_{2,\xi}\right]\eta\\ &\quad-\left[\eta\left(\varphi_{1,\xi\eta}-\varphi_{2,\xi\xi}\right)+\tfrac{\kappa+2}{2}\varphi_{1,\xi}-\tfrac{\kappa-4}{2}\varphi_{2,\eta}\right]\xi\\ &\quad+\tfrac{\eta_1^2-\eta}{\xi^2+\eta^2}\left[\left(\varphi_{1,\eta}-\varphi_{2,\xi}\right)\eta-\left(\varphi_{1,\xi}+\varphi_{2,\eta}\right)\xi\right], \end{aligned} \tag{21}$$

The boundary conditions (4), (5) are satisfied if

$$\varphi_i = \sum_{n=1}^{\infty}\varphi_{in}, \quad i=1,2, \tag{22}$$

where

$$\begin{aligned} \varphi_{1n} &= A_{1n}\sinh\left(n\eta\right)\sin\left(n\xi\right),\\ \varphi_{2n} &= A_{2n}\cosh\left(n\eta\right)\cos\left(n\xi\right). \end{aligned}$$

By inserting (22) in (20) and (21) we will receive the following expressins for the displacements:

$$
\begin{aligned}
\bar{u} =&\sum_{n=1}^{\infty}\{-[n\eta\xi\cosh(n\eta)(A_{1n}+A_{2n})+(\kappa-1)\xi\sinh(n\eta)A_{1n}]\sin(n\xi)\\
&+\left[n\eta_1^2\sinh(n\eta)(A_{1n}+A_{2n})-(\kappa-1)\eta\cosh(n\eta)A_{2n}\right]\cos(n\xi)\},\\
\bar{v} =&\sum_{n=1}^{\infty}\{\left[n\eta_1^2\cosh(n\eta)(A_{1n}+A_{2n})+(\kappa-1)\eta\sinh(n\eta)A_{1n}\right]\sin(n\xi)\\
&+[n\eta\xi\sinh(n\eta)(A_{1n}+A_{2n})-(\kappa-1)\xi\cosh(n\eta)A_{2n}]\cos(n\xi)\},
\end{aligned}
\tag{23}
$$

and for the stresses the following:

$$
\begin{aligned}
\frac{h_0^2}{2\mu}\sigma_{\eta\eta} =& \sum_{n=1}^{\infty}\left\{\left[n^2\eta_1^2\sinh(n\eta)(A_{1n}+A_{2n})\right.\right.\\
&\left.+n\eta\cosh(n\eta)\left(\tfrac{\kappa}{2}A_{1n}-\tfrac{\kappa-2}{2}A_{2n}\right)\right]\sin(n\xi)\\
&+\left[n^2\eta\xi\cosh(n\eta)(A_{1n}+A_{2n})\right.\\
&\left.+n\xi\sinh(n\eta)\left(\tfrac{\kappa-2}{2}A_{1n}-\tfrac{\kappa}{2}A_{2n}\right)\right]\cos(n\xi)\\
&-\tfrac{\eta_1^2-\eta^2}{\xi^2+\eta^2}\left[n\eta\cosh(n\eta)(A_{1n}+A_{2n})\sin(n\xi)\right.\\
&\left.\left.-n\xi\sinh(n\eta)(A_{1n}+A_{2n})\cos(n\xi)\right]\right\},\\
\frac{h_0^2}{2\mu}\tau_{\xi\eta} =& \sum_{n=1}^{\infty}\left\{\left[n^2\eta_1^2\cosh(n\eta)(A_{1n}+A_{2n})\right.\right.\\
&\left.+n\eta\sinh(n\eta)\left(\tfrac{\kappa-2}{2}A_{1n}-\tfrac{\kappa}{2}A_{2n}\right)\right]\cos(n\xi)\\
&-\left[n^2\eta\xi\sinh(n\eta)(A_{1n}+A_{2n})\right.\\
&\left.+n\xi\cosh(n\eta)\left(\tfrac{\kappa}{2}A_{1n}-\tfrac{\kappa-2}{2}A_{2n}\right)\right]\sin(n\xi)\\
&-\tfrac{\eta_1^2-\eta^2}{\xi^2+\eta^2}\left[n\xi\cosh(n\eta)(A_{1n}+A_{2n})\sin(n\xi)\right.\\
&\left.\left.+n\eta\sinh(n\eta)(A_{1n}+A_{2n})\cos(n\xi)\right]\right\},\\
\frac{h_0^2}{2\mu}\sigma_{\xi\xi} =& \sum_{n=1}^{\infty}\left\{-\left[n^2\eta_1^2\sinh(n\eta)(A_{1n}+A_{2n})\right.\right.\\
&\left.+n\eta\cosh(n\eta)\left(\tfrac{\kappa-4}{2}A_{1n}-\tfrac{\kappa+2}{2}A_{2n}\right)\right]\sin(n\xi)\\
&-\left[n^2\eta\xi\cosh(n\eta)(A_{1n}+A_{2n})\right.\\
&\left.+n\xi\sinh(n\eta)\left(\tfrac{\kappa+2}{2}A_{1n}-\tfrac{\kappa-4}{2}A_{2n}\right)\right]\cos(n\xi)\\
&+\tfrac{\eta_1^2-\eta^2}{\xi^2+\eta^2}\left[n\eta\cosh(n\eta)(A_{1n}+A_{2n})\sin(n\xi)\right.\\
&\left.\left.-n\xi\sinh(n\eta)(A_{1n}+A_{2n})\cos(n\xi)\right]\right\}.
\end{aligned}
\tag{24}
$$

Instead of conditions (3) we have to take their equivalent following expressions

$$
\begin{aligned}
\tfrac{1}{2\mu}\left(\sigma_{\eta\eta}\cdot\eta_1-\sigma_{\xi\eta}\cdot\xi\right) &= -\eta_1\left(\varphi_{1,\xi\xi}+\varphi_{2,\xi\eta}\right)-\tfrac{\kappa}{2}\varphi_{1,\eta}-\tfrac{\kappa-2}{2}\varphi_{2,\xi},\\
\tfrac{1}{2\mu}\left(\sigma_{\eta\eta}\cdot\xi+\sigma_{\xi\eta}\cdot\eta_1\right) &= \eta_1\left(\varphi_{1,\xi\eta}-\varphi_{2,\xi\xi}\right)+\tfrac{\kappa-2}{2}\varphi_{1,\xi}-\tfrac{\kappa}{2}\varphi_{2,\eta}.
\end{aligned}
\tag{25}
$$

The constants A_{1n} and A_{2n} are defined if the boundary conditions (3) are satisfied. Thus, from the (3), (6) (22), (25) we receive following equations:

$$\begin{gathered}\left[\left(n^2\eta_1 \sinh\left(n\eta_1\right) - n\tfrac{\kappa}{2}\cosh\left(n\eta_1\right)\right) A_{1n}\right.\\ \left.+\left(n^2\eta_1 \sinh\left(n\eta_1\right) + n\tfrac{\kappa-2}{2}\cosh\left(n\eta_1\right)\right) A_{2n}\right] = \tilde{F}_{1n},\\ \left[\left(n^2\eta_1 \cosh\left(n\eta_1\right) + n\tfrac{\kappa-2}{2}\sinh\left(n\eta_1\right)\right) A_{1n}\right.\\ \left.+\left(n^2\eta_1 \cosh\left(n\eta_1\right) - n\tfrac{\kappa}{2}\sinh\left(n\eta_1\right)\right) A_{2n}\right] = \tilde{F}_{2n},\\ n = 1, 2, \ldots\end{gathered} \tag{26}$$

where $\tilde{F}_{1n}$ and $\tilde{F}_{2n}$ are the coefficients of expansion into Fourier series $f_1(\xi) = \sum\limits_{n=1}^{\infty} \tilde{F}_{1n} \sin(n\xi)$ and $f_2(\xi) = \sum\limits_{n=1}^{\infty} \tilde{F}_{2n} \cos(n\xi)$, respectively, $f_1(\xi) = P\eta_1 / \left(\xi^2 + \eta_1^2\right)$ and $f_2(\xi) = P\xi / \left(\xi^2 + \eta_1^2\right)$ functions.

As seen, the main matrix of the system (26) has a block-diagonal form. The dimension of each block is 2×2 and the determinant is not equal to zero, but in infinite the determinant of block strive to the finite number different to zero. Thus, two equations will be solved, to two unknowns A_{1n} andA_{2n}. After solving this system, we find coefficients A_{1n} andA_{2n}, and put them into formulas (23) and (24), we get displacements and stresses at any points of the body.

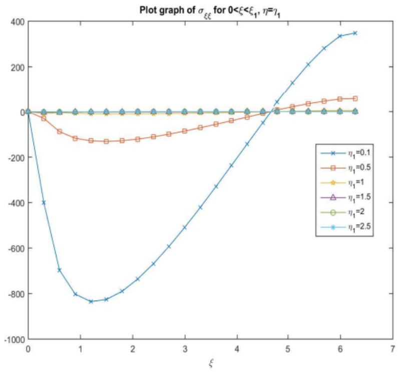
It is very easy to establish convergence of (23), (24) functional series on the area $\bar{D}_1 = \{-\xi_1 \le \xi \le \xi_1,\ 0 \le \eta \le \eta_1\}$ by construction of the corresponding uniform convergent numerical majorizing series. So we have the following

Proposal 1. The functional series corresponding to (23), (24) are absolutely and uniform by convergent series on the area $\bar{D}_1 = \{-\xi_1 \le \xi \le \xi_1,\ 0 \le \eta \le \eta_1\}$.

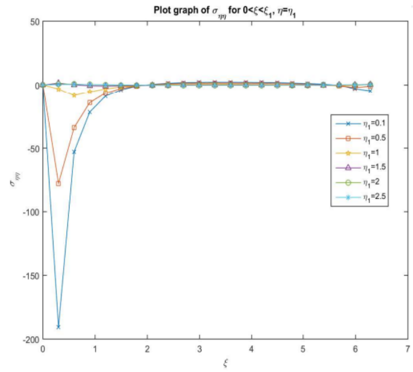
4 Computer implementation

Numerical results are obtained at points of the body (see. Fig. 1) for the following data: $\nu = 0.3$, $E = 2 * 10^6 kg/cm^2$, $P = -10 kg/cm^2$, $\eta_1 = 2$, $\xi_1 = 2\pi$. Numerical calculations and construction relevant 2D and 3D graphs are made by MATLAB's software.

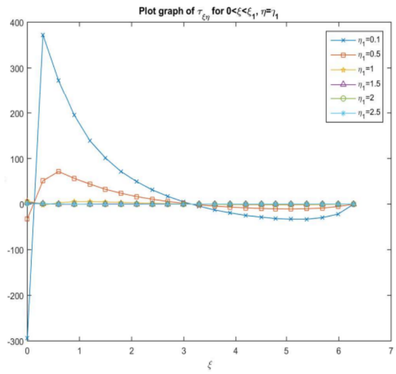
Figures 3 show the graphs of $\sigma_{\xi\xi}$, $\sigma_{\eta\eta}$, $\tau_{\xi\eta}$ stresses and u, v displacements, respectively on the line $\eta = \eta_1$, when (3), (4), (5) boundary conditions are valid and normal stress is applied to the parabolic boundary, while tangential stress equals zero. Figure 4 show the distribution of stresses and displacements in the region bounded by curved lines $\eta = \eta_1$ and $\xi = \xi_1$.



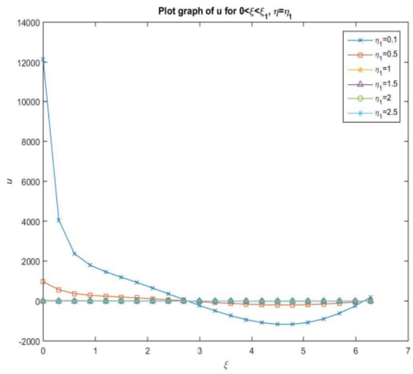
(a)



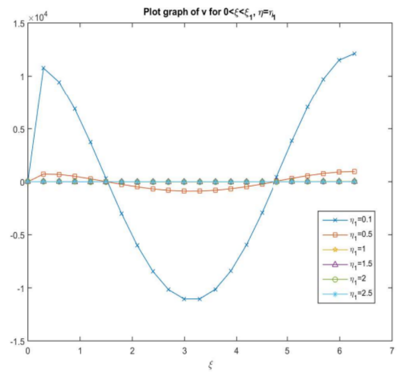
(b)



(c)

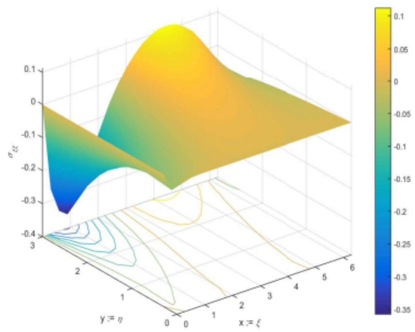


(d)

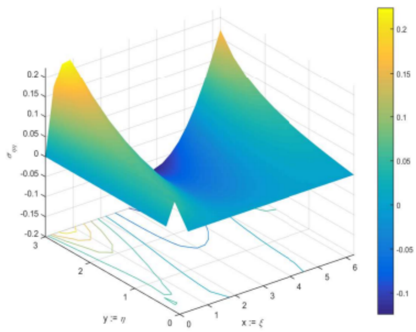


(e)

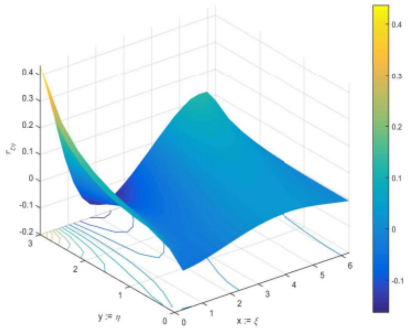
Figure 3: Schedules of (a) normal $\sigma_{\eta\eta}$, (b) shearing $\sigma_{\xi\xi}$, (c) tangential $\tau_{\xi\eta}$ stresses and (d) normal u, (e) tangential v displacements in point $M\,(\xi, \eta_1)$ for $\eta_1 = 0.1;\ 0.5;\ 1;\ 1.5;\ 2;\ 2.5$, when $0.1 \le \xi \le \xi_1$.



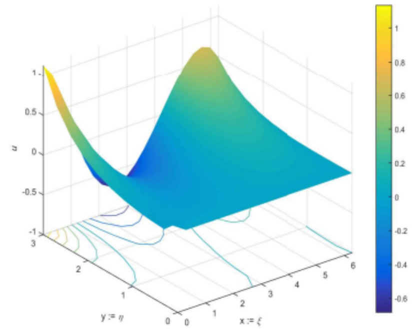
(a)



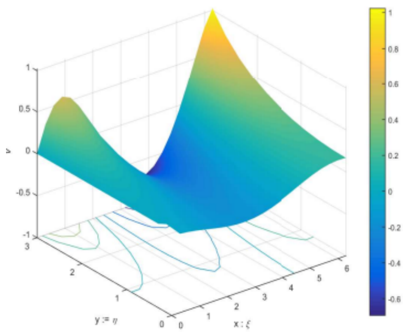
(b)



(c)



(d)



(e)

Figure 4: Distribution of (a) normal $\sigma_{\eta\eta}$, (b) shearing $\sigma_{\xi\xi}$, (c) tangential $\tau_{\xi\eta}$ stresses and (d) normal u, (e) tangential v displacements in the region bounded by curved lines $\eta = \eta_1$ and $\xi = \xi_1$.

5 Conclusion

The main results of this work can be formulated as follows.

- ◇ The equilibrium equations are written in terms of parabolic coordinates.
- ◇ The solution of the equilibrium equation (1) is obtained by the method of separation of variables. The solution is constructed using its general representation by two harmonic functions.
- ◇ In the parabolic coordinates exact solution of two-dimensional static boundary value problem for the elasticity is constructed for the homogeneous isotropic body occupying the domain bounded by coordinate lines of parabolic coordinates.
- ◇ 2D and 3D graphs for some numerical values of BVP are given and considered. The graphs are obtained by using computer software MATLAB.

Appendix A.
Some main expressions in parabolic coordinates

In orthogonal parabolic coordinate system ξ, η $(-\infty < \xi < \infty,\ 0 \leq \eta < \infty)$ [22, 23] we have

$$h_\xi = h_\eta = h = c\sqrt{\xi^2 + \eta^2}, \qquad x = \frac{c}{2}\left(\xi^2 - \eta^2\right), \quad y = c\xi\eta,$$

where $h_\xi,\ h_\eta$ are Lame's coefficients of the system of parabolic coordinates, c is a scale coefficient, $x,\ y$ are Cartesian coordinates.

The coordinate axis are parabolas

$$\begin{array}{ll} y^2 = -2c\xi_0^2\left(x - \frac{c\xi_0^2}{2}\right), & \xi_0 = const, \\ y^2 = -2c\eta_0^2\left(x + \frac{c\eta_0^2}{2}\right), & \eta_0 = const. \end{array}$$

Laplace's equation $\Delta f = 0$, where $f = f(\xi, \eta)$, in the parabolic coordinates has the form

$$\frac{1}{c^2\left(\xi^2 + \eta^2\right)}\left(f_{,\xi\xi} + f_{,\eta\eta}\right) = 0.$$

We have to find solution of the equation in following form

$$f = X(\xi) \cdot Y(\eta),$$

Then by separation of variables we will receive.

$$\frac{1}{c^2\left(\xi^2 + \eta^2\right)}\left[\frac{X''}{X} + \frac{Y'}{Y}\right] = 0.$$

From here

$$X'' + mX = 0,$$
$$Y'' - mY = 0,$$

where m any constant, their solutions are [24]

$$X = C_1 \cos(m\xi) + C_2 \sin(m\xi),$$
$$Y = C_3 e^{m\eta} + C_4 e^{-m\eta} = C_3^* \cosh(m\eta) + C_4^* \sinh(m\eta).$$

So

$$f(\xi, \eta) = \left(C_3 e^{m\eta} + C_4 e^{-m\eta}\right)\left(C_1 \cos(m\xi) + C_2 \sin(m\xi)\right)$$

or

$$f(\xi, \eta) = \left(C_3^* \cosh(m\eta) + C_4^* \sinh(m\eta)\right)\left(C_1 \cos(m\xi) + C_2 \sin(m\xi)\right).$$

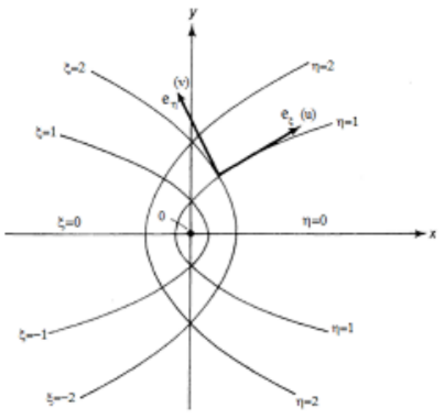


Figure 5: Parabolic coordinate system.

e_ξ, e_η are unit vectors.

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