TO UNIFIED SYSTEM OF EQUATIONS OF CONTINUUM MECHANICS AND SOME MATHEMATICAL PROBLEMS FOR THIN-WALLED STRUCTURES

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Abstract

Proposed are the unified mathematical models of continuum mechanics containing in particular Navier-Stokes, Euler Differential equations, systems of PDEs of Solid mechanics. In the second part, the method of constructing 2D nonlinear models of von Kármán-Mindlin-Reissner type for a binary mixture of porous, piezo and viscous elastic thin-walled structures with variable thickness is given.

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1 Unified Form for Some Nonlinear Problems of Continuum Mechanics

Within Newtonian mechanics and Noll's axiomatic, we propose a unified dynamic system of pseudo-differential equations. The such form allows us to prove that the nonlinear phenomena observed in problems of solid mechanics can also be detected in Navier-Stokes type equations, and vice versa. For this case the basic system of PDE has the following form:

$$\rho\frac{D_\Gamma^2 u}{Dt^2} = f-(1-\Gamma)\nabla p+\nabla[(1+\nabla u)\tau], \quad \frac{D_\Gamma^2 u}{Dt^2} = \begin{cases}\partial^2 u/\partial t^2, & \Gamma = 1\\ Dv/Dt, & \Gamma = 0\end{cases}, \quad (1.1)$$

where ρ is a density, p is pressure, f is known volume force, D/Dt is a total or convective derivative, τ is a stress tensor, $u = (u_1, u_2, u_3)^T$ and $(v_1, v_2, v_3)^T$ denote displacement and velocity vectors.

Newton's type law for viscous flow and Hooke's generalized law for solid structures may be recovered from the following expression:

$$\tau = \Big[(1-\Gamma)\frac{\partial}{\partial t} + \Gamma\Big] A_\Gamma \cdot \varepsilon \tag{1.2}$$

Here the symmetric matrix A_Γ corresponds to fluid if $\Gamma = 0$ and to solid media if $\Gamma = 1$. The strain tensor is $\varepsilon = (\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{23}, \varepsilon_{13}, \varepsilon_{12})^T$, where $2\varepsilon_{ij} = \partial_i u_j + \partial_j u_i + u_{k,i}u_{k,j}$.

For conditions of conservation of mass or equations of continuity we have:

$$[(1-\Gamma)\partial_t + \Gamma] B_\Gamma[\varepsilon] = 0. \tag{1.3}$$

where $B_0[\rho, \varepsilon] = \partial_t \rho + \nabla(\rho\nu), B_1[\varepsilon] = (B_{11}, B_{12}, B_{13}, B_{14}, B_{15}, B_{16})^T$ describes the St.Venant-Beltrami conditions:

$$\begin{aligned} &B_{1i}(\varepsilon) = \varepsilon_{ii,kl} + \varepsilon_{kl,ii} - \varepsilon_{li,kl} - \varepsilon_{ki,li} + C_{1i}(u), \\ &i,k,l = 1,2,3,\ k \neq l,\ i \neq k,\ i \neq l \\ &B_{1,7-i}(\varepsilon) = \varepsilon_{ii,kl} + \varepsilon_{kl,ii} - \varepsilon_{li,ki} - \varepsilon_{ki,li} + C_{1,7-i}(u), \\ &i = 1,2,3,\ k = l = i+1,\ x_1 = x_4, \end{aligned} \tag{1.4}$$

$C_{ij}(u)$ are nonlinear differential forms of at most 3 orders. In the classical case, when

$$\varepsilon_{\alpha\alpha} = u_{\alpha,\alpha} + 0.5u_{3,1}^2,$$

$$\varepsilon_{12} = 0.5(u_{1,2} + u_{2,1} + u_{3,1}u_{3,2}),\ \ C_{11}(u) = 2(u_{3,12})^2 - 2u_{3,11}u_{3,22},$$

corresponds to the well-known Monge-Ampére form: $[u, \varphi] = \partial_{11}u\partial_{22}\varphi - 2\partial_{12}u\partial_{12}\varphi + \partial_{22}u\partial_{11}\varphi$.

For completeness to the general presentation (1.1)-(1.3) must be added the energy conservation (energy balance) equations which have evidently the same with (1.3) form. Now we consider this problem when continuum media is anisotropic elastic structures characterized by 13 independent constants.

If $\Gamma = 1$ and $\nabla(1 + \nabla u)\tau = (\partial_j(\tau_{ij} + \tau_{kj}u_{1,k}), \partial_j(\tau_{2j} + \tau_{kj}u_{2,k}), \partial_j(\tau_{3j} + \tau_{kj}u_{3,k})$ from (1.1) follows the system of nonlinear PDE of the spatial theory of elasticity:

$$\rho\partial_{tt}u_i = -f_i + \partial_j(\tau_{ij} + \tau_{kj}u_{i,k}). \tag{1.5}$$

See for example [1].

Hooke's generalized law ($\varepsilon = A\tau, A_1 \overset{def}{\equiv} A$) has the form:

$$\varepsilon_{ii} := u_{i,i} = a_{ij}\tau_{jj} + a_{i6}\tau_{12},\ \ \varepsilon_{a3} := 0.5(u_{\alpha,3} + u_{3,\alpha})$$

$$= a_{a+3,a+3}\tau_{\alpha 3} + a_{\alpha+3,6-\alpha}\tau_{33-\alpha},$$

$$\varepsilon_{12} := 0.5(u_{1,2} + u_{2,1}) = a_{6i}\tau_{ii} + a_{66}\tau_{12}.$$

Here A is the symmetric matrix and the compliance constant a_{ij} may be written in terms of engineering coefficients [2,3]:

$$a_{ii} = 1/E_i, \;\; a_{ij} = a_{ji} = -\nu_{ij}/E_i = -\nu_{ji}/E_i,$$

$$a_{6-\alpha,6-\alpha} = 1/E_{3+\alpha,3+\alpha}, \;\; a_{66} = 1/E_{66},$$

$$a_{3+\alpha,6-\alpha} = -\nu_{3+\alpha,6-\alpha}/E_{3+\alpha}, \;\; a_{i6} = a_{6i} = -\nu_{i6}/E_i = -\nu_{6i}/E_6.$$

Here according to [3], E_i are 6 "true modulus of rigidity", ν_{ij} are 7 distributors of rigidity and the angles of turn absent, or E_i are Young's type modulus of elasticity, ν_{ij} are the ratios of Poisson.

For potential energy $(A\tau,\tau)$ we have the following inequality:

$$(A\tau,\tau) = \sum_{ij}(A\tau)_{ij}\tau_{ij} \geq \frac{1}{E_i}(1 - \nu_{i1} - \nu_{i2} - \nu_{i6})\tau_{ii}^2$$

$$+\frac{1}{E_{6-\alpha}}(1 - \nu_{3+\alpha,6-\alpha})\tau_{3-\alpha 3}^2 + \frac{1}{E_6}(1 - \nu_{6i} signi)\tau_{12}^2.$$

Thus $(A\tau,\tau)$ is positive definiteness, if

$$(\nu_{i1} + \nu_{i2} + \nu_{i6} < 1), \;\; i = 1,2,3,6.$$

For an orthotropic case $\nu_{3+\alpha,6-\alpha} = \nu_{i6} = 0$ (there are 10 independent coefficients) and the above inequality is obviously satisfied as $\nu_{ij} < 0.5, \forall_{ij}$.

Further, if $\Gamma = 0, \nabla u = 0, \tau_{ij} = -\frac{2}{3}\mu\delta_{ij} divv + \mu(v_{ij} + v_{ji})$, δ_{ij} is Kronecker delta, μ is dynamical viscosity and a fluid is Newtonian liquid, then it's evident that from (1.1) follows: the Euler equations

$$\rho\lfloor \partial v_i/\partial t + v_i v_{i,j}\rfloor = -p_{,i} - \delta_{i3}\mathrm{g}, \quad i = 1,2,3. \tag{1.6}$$

For clearing, we remind that for $i = 1$, from (1.6) follow:

$$\rho\Big[\frac{\partial v_1}{\partial t} + v_1\partial_1 v_1 + \nu_2\partial_2 v_1 + v_3\partial_3 v_1\Big] = -\partial_1 p, \; \partial_i v_j = \frac{\partial v_j}{\partial x_i}, \; \delta_{13} = 0,$$

Navier - Stokes' PDEs

$$\rho\Big[\partial v_i/\partial t + v_j v_{i,j}\Big] = -p_{,i} - f_i + \tau_{ij,j}. \tag{1.7}$$

See, for example [4].

If $\Gamma = 0$ and $\nabla u \neq 0$, (1.1) represents Navier - Stokes type (new class) PDEs.

Our above elaborations are in conformity Newton's second law: $\rho\partial_t v = -f$ and different from it with concrete substance.

In case if on some continuum media acts also the electro-magnetic field (with PDEs (1.1)-(1.3)) using methodology of works [5, 6] it's necessity to consider Maxwell' dynamical system. For example, for them we use a sufficient convenience form [6, p.46]:

$$\varepsilon_{ijk}E_{j,k} + B_{j,t} = 0,\ \varepsilon_{ijk}H_{j,k} - Di, t = 0_i,\ B_{i,i} = 0, \qquad (1.8em)$$

where ε_{ijk} is an antisymmetric unit tensor, $E,\ H$ are tensions of an electric and magnetic fields, $D,\ B$ are the electrical and a magnetic induction vectors. Further, we also have:

$$E_{k,ki} - E_{i,kk} = -\mu_0 D_{i,tt},\ \ (\nu_0 \text{ is a magnetic penetrance}), \qquad (1.9em)$$

Thus, the systems (1.1), (1.3), (1.8em), (1.9em), (1.2) (which must be corrected by state equations type of (1.45) [1], p. 24) and the corresponding initial and boundary value conditions are presenting mathematical models for problems, connected with an emission of an electromagnetic waves of piezo-electric and electrically conductive continuum media.

I underline that in these monographs [6, 7] or other ones, were considered and investigated linear cases for corresponding thermo-dynamical problems and as the basic equation for the electro-magnetic elastic plate and shell was used only Kirchhoff-Love' classical theory for isotropic media.

For completeness, in addition to (1.1)-(1.3), the energy conservation (energy balance) equalities, which have evidently the same form as (1.3), must be considered. From (1.1) Euler and Navier-Stokes' PDEs (see [4]) can be recovered if

$$\Gamma = 0,\ \nabla u = 0,\ \tau_{ij} = -\frac{2}{3}\mu\delta_{ij} divv + \mu(v_{i,j} + v_{j,i}).$$

If $\Gamma = 1$ and $\nabla(1 + \nabla u)\tau = (\partial_j(\tau_{1j} + \tau_{kj}u_{1,k}), \partial_j(\tau_{2j} + \tau_{kj}u_{2,k}), \partial_j(\tau_{3j} + \tau_{kj}u_{3,k}))^T$ from (1.1) follows the system of nonlinear PDEs of the spatial theory of elasticity (see for example. [8]); If $\Gamma = 0$ and $\nabla u \neq 0$, (1.1) represents Navier - Stokes type (new class) PDEs.

2 Some Mathematical Problems of Thin Walled Structures

One of the most principal objects in development of mechanics and mathematics is a system of nonlinear differential equations for the elastic isotropic plate constructed by von Kármán. In 1978 Truesdell expressed a doubt:

"Physical Soundness" of von Kármán system. This circumstance generated the problem of justification of von Kármán system. Afterwards this problem is studied by many authors, but with most attention it was investigated by Ciarlet. In particular, he wrote: "The von Kármán equations may be given a full justification by means of the leading term of a formal asymptotic expansion" ([8], p. 368). This result obviously is not sufficient for a justification of the "Physical Soundness" of this system, because representations by asymptotic expansions is dissimilar and leading terms are only coefficients of power series without any "Physical Soundness."

Based on [9], the method of constructing such anisotropic nonhomogeneous 2D nonlinear models of von Kármán Mindlin-Mindlin-Reissner (KMR) type for binary mixtures; (poro/visco/piezo-electric/electrically conductive) elastic thin-walled structures with variable thickness is given, by means of which the terms become physically sound. The corresponding variables are quantities with certain physical meaning: averaged components of the displacement vector, bending and twisting moments, shearing forces, rotation of normals, surface efforts. The given method differs from the classical one by the fact that according to the classical method, one of the equations of von Kármán system represents one of Saint-Venant's compatibility conditions, i.e. it's obtained on the basis of geometry and not taking into account the equilibrium equations.

At last we remark that in dynamical cases the corresponding system contains wave processes not only in the vertical, but also in the horizontal direction. The corresponding equations are [9]:

$$\begin{aligned}
&(D\Delta^2 + 2h\rho\partial_{tt} - 2DE^{-1}(1+\nu)\rho\partial_{tt}\Delta)w \\
&= \Big(1 - \tfrac{h^2(1+2\gamma)(2-\nu)}{3(1-\nu)}\Delta\Big)(\mathrm{g}_3^+ - \mathrm{g}_3^-) \\
&+2h\Big(1 - \tfrac{h^2(1+2\gamma)}{3(1-\nu)}\Delta\Big)[w,\varphi] + h(\mathrm{g}_{\alpha\alpha}^+ - \mathrm{g}_{\alpha\alpha}^-) \\
&-\int_{-h}^{+h}\Big(tf_{\alpha\alpha} - \Big(1 - \tfrac{1}{1-\nu}\Delta(h^2-t^2)\Big)f_3\Big)dt
\end{aligned} \tag{2.1}$$

$$\begin{aligned}
&\Big(\Delta^2 - \tfrac{1-\nu^2}{E}\rho\Delta\partial_{tt}\Big)\varphi = -\tfrac{E}{2}[w,w] \\
&+\tfrac{\nu}{2}\Big(\Delta - \tfrac{2\rho}{E}\partial_{tt}\Big)(\mathrm{g}_3^+ + \mathrm{g}_3^-) + \tfrac{1+\nu}{2h}f_{\alpha\alpha}
\end{aligned} \tag{2.2}$$

From (2.1), (2.2) we get von Kármán system (in the dynamical case too) if $\gamma = -0.5, \rho = \mathrm{g}^{\pm} = f_\alpha = 0$. Thus the von Kármán classical system gives the possibility to use methods of Harmonic Analysis. Since the new dynamical terms are $\Delta\partial_{tt}\varphi$ and also $\partial_{tt}(\mathrm{g}_3^+ + \mathrm{g}_3^-)$, therefore the KMR type (2.1)-(2.2) systems describe new nonlinear wave processes. We remark that if the equations (2.1), (2.2) are in the final form it's evident that for them it is not possible to apply the direct Fourier Analysis technique. Because this system is nonlinear and both DEs contain dynamical members against von Kármán equations in the classical form.

In addition, an equation corresponding to (2.2) by von Kármán, A. Föppl, Love, Lukasievicz, Timoshenko, Donnel, Lev Landau, Ciarlet, Antmann et al. were constructed by the condition $\varepsilon_{11,22}-2\varepsilon_{12,12}+\varepsilon_{22,11}=-0.5[u_3,u_3]$ and Hooke's law (but without using the equilibrium equations!). As we prove in works [9] the form (2.2) follows immediately for more general cases, when thin-walled elastic structures are anisotropic and if we use Hooke's law, equilibrium equations with and nonlinear relations between strain tensor and displacement vector:

$$\varepsilon_{\alpha,\beta}=0.5(u_{\alpha,\beta}+u_{\beta,\alpha}+u_{3,\alpha}u_{3,\beta}).$$

Now we prove that (2.2) equations in the dynamical case has the following form:

$$\Big(-\frac{1-\nu^2}{E}\rho_1\Delta\partial_{tt}\Big)\Phi=\frac{\nu}{2}\Big(\Delta-\frac{2\rho_1}{E}\partial_{tt}\Big)(\mathrm{g}_3^{+}+\mathrm{g}_3^{-})+\frac{1+\nu}{2h}f_{\alpha,\alpha}. \tag{2.3}$$

Thus we must demonstrate that both way give the expression $\Delta^2\Phi$ $-0.5E[w,w]$. In fact, we constructed (2.2) by using the following expression (see [10]) :

$$\begin{cases}(\lambda^*+2\mu)\Delta(\bar\varepsilon_{11}+\bar\varepsilon_{22})=\\ (2\mu(3\lambda+2\mu))^{-1}(\lambda+2\mu)(\lambda^*+2\mu)\Delta(\bar\sigma_{11}+\bar\sigma_{22})+\cdots\\ =\mu((-1)^{\alpha+\beta}\partial_{3-\alpha}\partial_{3-\beta}\bar u_{3,\alpha}\bar u_{3,\beta})+\cdots,\end{cases} \tag{2.4}$$

where dots denote other different members from (2.2). Let us $\bar\sigma_{\alpha,\beta}=(-1)^{\alpha+\beta}\partial_{3-\alpha}\partial_{3-\beta}\Phi$, then from the preliminary equation follows (2.2) or: $\Delta^2\Phi=-0.5E[w,w]+\cdots$ From St. Venant-Beltrami compatibility conditions it is evident that

$$\Delta^2\Phi+0.5E[w,w]\equiv 0.$$

The mathematical models considered in [9] , ch.I contain a new quantity, which describes an effect of boundary layer. Existence of this member not only explains a set of paradoxes in the two-dimensional elasticity theory (Babushka, Lukasievicz, Mazia, Saponjan), but also is very important for example for process of generating cracks and holes (details see in [9], ch.1, 3.3). Further, let us note that in [10] equations of (2.2) type are constructed with respect to certain components of stress tensor by differentiation and summation of two differential equations. Also other equations of KMR type, which differ from (2.2) type equation, are equivalent to the system, where the order of each equation is not higher than two. For example, in the isotropic case, obviously, for coefficients we have [10]: $c_{\alpha,\alpha} = \lambda^* + 2\mu, c_{66} = 2\mu, c_{12} = \lambda^*, c_{\alpha,6} = 0, \lambda^* = 2\lambda\mu(\lambda+2\mu)^{-1}$, λ and μ are the Lame coefficients. Then the system (2.1) of [10] is presented in a form:

$$\begin{aligned}&(\lambda^* + 2\mu)\partial_1\tau + \mu\partial_2\omega \\ &= \tfrac{1}{2h}\bar{f}_1 + \mu(\partial_2(\bar{u}_{3,2})^2 - \partial_2(\bar{u}_{3,1}\bar{u}_{3,2})) - \tfrac{\lambda}{2h(\lambda+2\mu)}(\sigma_{33,1}, 1),\end{aligned} \tag{2.5a}$$

$$\begin{aligned}&\mu\partial_1\omega + (\lambda^* + 2\mu)\partial_2\tau \\ &= \tfrac{1}{2h}\bar{f}_2 + \mu(\partial_2(\bar{u}_{3,1})^2 - \partial_1(u\bar{_{3,1}}\bar{u}_{3,2})) - \tfrac{\lambda}{2h(\lambda+2\mu)}(\sigma_{33,2}, 1),\end{aligned} \tag{2.5b}$$

where the functions: $\tau = \bar{\varepsilon}_{\alpha,\alpha}, \omega = \bar{u}_{1,2} - \bar{u}_{2,1}$ correspond to plane expansion and rotation respectively. Thus, in the dynamical case the KMR type systems are (2.1) and (2.2). In the statical case from (2.5) immediately follows such relations:

$$\frac{\nu}{2}\Delta(\mathrm{g}_3^+ + \mathrm{g}_3^-) + \frac{1+\nu}{2h}f_{\alpha,\alpha} = 0.$$

In general this relation is not true or if it is true then these expressions are consequences of compatability conditions (see p.204, [1])

$$\iiint_{\Omega_h} f d\omega + \iint_{S+S^\pm} \mathrm{g} ds = 0.$$

The system (2.1)-(2.2) represents 2Dim mathematical model having clear practical meaning where it is possible to consider together the methods of Mathematical Analysis (in wide sense) and Optimal Control Theory of Bellman-Pontryagin.

Let us consider the main terms from (2.1):

$$D'\Delta[w,\varphi] = D'([\Delta w, \varphi] + [w, \Delta\varphi] + 2[\partial_\alpha w, \partial_\alpha \varphi])$$

$$(D' = 4h^3(1+2\gamma)/3(1-\nu)),\ D\Delta^2 w.$$

By using for simplicity the typical relations as $\partial_{11}\varphi = \bar{\sigma}_{22}, \partial_{12}\varphi = -\bar{\sigma}_{12},\ \partial_{22}\varphi = \bar{\sigma}_{11}$, the first term may be rewritten in the following form:

$$\begin{aligned}\Delta[w,\varphi] = {} & (\bar{\sigma}_{11}\partial_{11}\Delta w + 2\bar{\sigma}_{12}\partial_{12}\Delta w + \bar{\sigma}_{22}\partial_{22}\Delta w) \\ & + (\partial_{11}w\Delta\bar{\sigma}_{11} + 2\partial_{12}w\Delta\bar{\sigma}_{12} + \partial_{22}w\Delta\bar{\sigma}_{22}) \\ & + 2(\bar{\sigma}_{11,\alpha}\partial_{11}w_{,\alpha} + 2\bar{\sigma}_{12,\alpha}\partial_{12}w_{,\alpha} + \bar{\sigma}_{22,\alpha}\partial_{22}w_{,\alpha}).\end{aligned} \tag{2.6}$$

The calculation and analysis by these expressions of a symbolical determinant show that the characteristic form of systems of type (2.1) and (2.2) may be positive, negative or zero numbers as well as an arbitrary continuous function of x, y. Other than that, in statical cases the system may have shock waves solutions. Here we must remark that $ED' = 4(1+2\gamma)(1+\nu)D$, as so if $\{f\}$ denotes physical dimension of value f, it's evident $\{\Delta^2 w\} = \{\Delta[w, \varphi/E]\}$.

Thus, the first summand of (2.6) may be defining the nonlinear wave processes for static cases. The structure of the third summand obviously corresponds to 2D soliton type solutions of Corteveg-de Vries or Kadomtsev-Petviashvili kind.

Analogous three-dimensional nonlinear model for anisotropic binary mixtures are presented in the works [11,12]. Here we generalize previously known model for poro-viscous-elastic binary mixtures. The constructed models together with certain independent scientific interest represent such form of spatial models, which allow not only to construct, but also to justify von KMR type systems as in the stationary, as well as in nonstationary cases. Under justification we mean assumption of "Physical Soundness" to these models in view of Truesdell-Ciarlet (see for example details in [9, ch.5]). As is known, even in case of isotropic elastic plate with constant thickness the subject of justification constituted an unsolved problem. The point is that von Kármán, Love, Timoshenko, Landau & Lifshits, Donell and others considered one of the compatibility conditions of Saint-Venant-Beltrami as one of the equations of the corresponding system of differential equations. This fact was verified also by Podio-Guidugli recently.

Further, let us note that in works [9] equations of (2.6) type are constructed with respect to certain components of the stress tensor by differentiation and summation of two differential equations. Besides other equations of KMR type, which differ from (2.5) type equation, are equivalent to the system, where the order of each equation is not higher than two.

We remind that matrices of compliability - A and rigidity - B in the formulae (1.5) [9] contain no more than thirteen independent elastic constants, i.e., at any point of body Ω_h even if one plane of elastic symmetry spreads, which is parallel to the coordinate plane-oxy. Using of formulae (2.25) [9], we have:

$$\sigma_{\alpha\alpha} = c_{\alpha\beta}\varepsilon_{\beta\beta} + c_{\alpha\alpha}\varepsilon_{12} + b_{13}b_{33}^{-1}\sigma_{33}, \quad \sigma_{\alpha 3} = c_{3-\alpha 3+\alpha}\varepsilon_{\alpha 3} + b_{45}\varepsilon_{\alpha 3-\alpha},$$

$$\sigma_{12} = c_{\alpha 6}\varepsilon_{\alpha\alpha} + c_{66}\varepsilon_{12} + b_{36}b_{33}^{-1}\sigma_{33}$$

where

$$c_{\alpha\alpha} = b_{\alpha\alpha} - b_{\alpha_3}^2 b_{33}^{-1}, \quad c_{12} = c_{21} = b_{12} - b_{13}b_{23}b_{33}^{-1},$$

$$c_{\alpha 6}=b_{\alpha 6}-b_{\alpha_3}b_{36}b_{33}^{-1},\ \ c_{66}=b_{66}-b_{36}^2b_{33}^{-1},$$

Then, obviously, from equations (1.1) and (1.2) (when $\Gamma=1$) follows:

$$2h\left[c_{\alpha\alpha}(\bar{u}_{\alpha,\alpha\alpha}+\tfrac{1}{2}\bar{A}_{\alpha\alpha,\alpha})+c_{\alpha 3-\alpha}(\bar{u}_{3-\alpha,12}+\tfrac{1}{2}\bar{A}_{3-\alpha 3-\alpha\alpha})\right.$$
$$\left.+\tfrac{1}{2}c_{\alpha 6}(\bar{u}_{\alpha,12}+\bar{u}_{3-\alpha,\alpha\alpha}+\bar{A}_{12,\alpha})\right]$$
$$+2h\left[c_{6\alpha}(\bar{u}_{\alpha,12}+\tfrac{1}{2}\bar{A}_{\alpha\alpha,3-\alpha})+\tfrac{1}{2}c_{66}(\bar{u}_{\alpha,3-\alpha 3-\alpha}+\bar{u}_{3-\alpha,12}+\bar{A}_{12,3-\alpha})\right]$$
$$+b_{33}^{-1}(b_{\alpha,3}\partial_\alpha+b_{36}\partial_{3-\alpha}\int\limits_{-h}^{h}\sigma_{33}dz=\int\limits_{-h}^{h}f_\alpha dz-\partial_\beta(\sigma_{k\beta}u_{\alpha,k})-(\mathrm{g}_\alpha^+-\mathrm{g}_\alpha^-),$$

$$A_{ij}=u_{k,i}u_{k,j}.$$

Further, from these equations, using the method ch. I [9], we have:

$$2h\left[\left(c_{\alpha\alpha}\partial_{\alpha\alpha}+\frac{3}{2}c_{\alpha 6}\partial_{12}+\frac{1}{2}c_{66}\partial_{3-\alpha 3-\alpha}\right)\bar{u}_\alpha+\left(\frac{1}{2}c_{66}\partial_{\alpha\alpha}+\right.\right.$$
$$\left.\left(c_{12}+\frac{1}{2}c_{66}\right)\partial_{12}+c_{3-\alpha 6}\partial_{3-\alpha 3-\alpha})\bar{u}_{3-\alpha}\right]$$
$$+h\left[\ (c_{\alpha\alpha}\partial_\alpha+c_{\alpha 6}\partial_{3-\alpha 3-\alpha})(u_{3,\alpha})^2\right.$$
$$\left.+(c_{\alpha 6}\partial_\alpha+c_{66}\partial_{3-\alpha})\bar{u}_{3,1}\bar{u}_{3,2}+(c_{12}\partial_\alpha+c_{3-\alpha 6}\partial_{3-\alpha})(\bar{u}_{3,3-\alpha})^2\ \right]$$
$$+b_{33}^{-1}(b_{\alpha 3}\partial_\alpha+b_{\alpha 6}\partial_3-\alpha)\int_{-h}^{h}\sigma_{33}dz=\bar{f}_\alpha. \tag{2.7}$$

Here

$$\bar{u}_\alpha=\frac{1}{2h}\int\limits_{-h}^{h}u_\alpha(x,y,z)dz,\ \ \bar{u}_3=\frac{3}{2h^3}\int\limits_{-h}^{h}(h^2-z^2)u_3(x,y,z)dz,$$

$$\bar{f}_\alpha=\int\limits_{-h}^{h}(f_\alpha-\partial_\beta(\sigma_{k\beta}u_{\alpha k}))dz-(\mathrm{g}_\alpha^+-\mathrm{g}_\alpha^-)-R_\alpha^{AN}$$
$$R_\alpha^{AN}=R_N^{AN}[u_1,u_2,u_3-\bar{u}_3]=\int\limits_{-h}^{h}(c_{\alpha\alpha}\partial_\alpha+c_{\alpha 6}\partial_{3-\alpha})(u_{k,\alpha}^2-(\bar{u}_{3,\alpha})^2).$$
$$+(c_{\alpha\alpha}\partial_\alpha+c_{66}\partial_{3-\alpha})(u_{k,1}u_{k,2}-\bar{u}_{3,1}\bar{u}_{3,2})$$
$$+(c_{12}\partial_\alpha+c_{3-\alpha 6}\partial_{3-\alpha})(u_{k,3-\alpha}^2-(u_{3,3-\alpha})^2)]dz$$

The system of equations (2.7), if we neglect the remainder terms R, for a linear case corresponds to the problem of defining generalized plane stress-stain state. For a nonlinear case from (2.7) it follows immediately one of the basic equations of the von Kármán system, corresponding to the Airy function if each equation is differentiated and summed (for details see below).

For an isotropic case, obviously, for coefficients we have $c_{\alpha,\alpha} = 2\nu, c_{66} = 2\mu,\ c_{12} = \lambda^*, c_{\alpha 6} = 0, \lambda^* = 2\lambda\mu(\lambda + 2\mu)^{-1},\ \lambda$ and μ are the Lame coefficients. Then the system (2.1) is presented in a form:

$$
\begin{aligned}
&(\lambda^* + 2\mu)\partial_1\tau + \mu\partial_2\omega \\
&= \frac{1}{2h}\bar{f}_1 + \mu(\partial_1(\bar{u}_{3,2})^2 - \partial_2(\bar{u}_{3,1}\bar{u}_{3,2})) - -\frac{\lambda}{2h(\lambda+2\mu)}\int_{-h}^{h}\sigma_{33,1}dz \\
&-\mu\partial_1\omega + (\lambda^* + 2\mu)\partial_2\tau \\
&= \frac{1}{2h}\bar{f}_2 + \mu(\partial_2(\bar{u}_{3,1})^2 - \partial_1(\bar{u}_{3,1}\bar{u}_{3,2})) - -\frac{\lambda}{2h(\lambda+2\mu)}\int_{-h}^{h}\sigma_{33,2}dz,
\end{aligned}
\tag{2.8}
$$

where the functions: $\tau = \bar{\varepsilon}_{\alpha,\alpha},\ \omega = \bar{u}_{1,2} - \bar{u}_{2,1}$ correspond to plane expansion and rotation. The second equation with respect to the Airy function the von Kármán system, following from (12) has the form:

$$
\begin{aligned}
&(\lambda^* + 2\mu)\Delta\bar{\varepsilon}_{\alpha,\alpha} = (\lambda^* + 2\mu)\Delta\left(\frac{1}{2\mu} - \frac{1}{\mu(3\lambda+2\mu)}\right)(\bar{\sigma}_{11} + \bar{\sigma}_{22}) \\
&= \mu(\partial_{11}(u_{3,2})^2 - 2\partial_{1,2}(\bar{u}_{3,1}\bar{u}_{3,2}) + \partial_{22}(\bar{u}_{3,1})^2) \\
&+\tfrac{1}{2h}\bar{f}_{\alpha,\alpha}\frac{1}{2h}\left(\frac{\lambda(\lambda^*+2\mu)}{2\mu(3\lambda+2\mu)} - \frac{\lambda}{\lambda+2\mu}\right)\int_{-h}^{h}\Delta\sigma_{33}dz.
\end{aligned}
$$

or

$$
\Delta(\bar{\sigma}_{11} + \bar{\sigma}_{22}) = -\frac{E}{2}[\bar{u}_3, \bar{u}_3] + \frac{\nu}{2h}\int_{-h}^{h}\Delta\sigma_{33}dz + \frac{1+\nu}{2h}f_{\alpha,\alpha} \tag{2.9}
$$

If we introduce the Airy function by a well-known way:

$$
\sigma_{\alpha,\beta} = (1)^{\alpha+\beta}\partial_{3-\alpha 3-\beta}\Phi,
$$

from (12) follows the second equation of the von Kármán system

$$
\Delta^2\Phi^* = -\frac{E}{2}[\bar{u}_3, \bar{u}_3] + \frac{\nu}{2}\Delta(\mathrm{g}_3^+ + \mathrm{g}_3^-) + \frac{1+\nu}{2h}f_{\alpha,\alpha} \tag{2.10}
$$

If we consider an orthotropic case, when $c_{\alpha,6} = 0$. Then from (2.7), obviously, it follows that

$$2h\left[c_{\alpha,\alpha}\partial_\alpha\bar{\varepsilon}_{\alpha,\alpha} + (c_{12}+c_{66})\partial_\alpha\bar{\varepsilon}_{3-\alpha 3-\alpha} + \frac{1}{2}(-1)^{3-\alpha}c_{66}\partial_{3-\alpha}(\bar{u}_{1,2}-\bar{u}_{2,1})\right]$$

$$+hc_{66}\left[\partial_{3-\alpha}(\bar{u}_{3,1}\bar{u}_{3,2}) - \partial_\alpha(\bar{u}_{3,2})^2\right] = \bar{f}_\alpha - b_{\alpha 3}b_{\alpha 3}^{-1}\int_{-h}^{h}\sigma_{33,\alpha}dz - R_\alpha^{AN}, \quad (2.11)$$

where

$$\bar{\varepsilon}_{\alpha,\alpha} = \frac{1}{2h}\int_{-h}^{h}(u_{\alpha,\alpha} + u_{k,\alpha}^2)dz.$$

When coefficients b and c satisfy the condition of generalized transversality ([9], p. 27), i.e. there are true the relations:

$$c_{11} = c_{22} = c_{12} + c_{66}, \quad b_{13} = b_{23},$$

then from (2.11) immediately it follows:

$$\begin{aligned}
&c_{11}\partial_1\tau + \frac{1}{2}c_{66}\partial_2\omega = \frac{1}{2h}\bar{f}_1 - b_{13}b_{33}^{-1}\frac{1}{2h}\int_{-h}^{h}\sigma_{33,1}dz \\
&-hc_{66}\left[\partial_2(\bar{u}_{3,1}\bar{u}_{3,2}) - \partial_1(\bar{u}_{3,2})^2\right] - R_1^{AN} \\
&c_{11}\partial_2\tau - \frac{1}{2}c_{66}\partial_1\omega = \frac{1}{2h}\bar{f}_2 - b_{23}b_{33}^{-1}\frac{1}{2h}\int_{-h}^{h}\sigma_{33,2}dz \\
&-hc_{66}\left[\partial_1(\bar{u}_{3,1}\bar{u}_{3,2}) - \partial_2(\bar{u}_{3,2})^2\right] - R_2^{AN}
\end{aligned} \quad (2.11a)$$

The systems of differential equations (2.7-2.10) obviously, are a splitting of one, corresponds to the function Φ^* from the von Kármán equations and equivalent to it in case differentiability of the functions $\bar{u}_\alpha$, which are averaged on a thickness of the plate of horizontal components of the displacement vector.

Thus, obtained the system of differential equations (2.7-2.10) is constructed from the initial three-dimensional problem of the theory of elasticity (1.1) - (1.5) [9]with respect to the averaged on a thickness of the components of the displacement vector-$\bar{u}$.

The another basic equation of the von Kármán system corresponds for a linear case to a bending problem. For clarity and completeness we now give a presentation of the second basic relation in case, when Ω_h is an isotropic elastic plate of constant thickness (more general case, when an elastic plate of a variable thickness with finite displacement is anisotropic and non-homogeneous see [9], ch.1).

$$\begin{aligned}
&\frac{(1-\nu)D}{2}\Delta u_\alpha^* + \frac{(1+\nu)D}{2}\partial_\alpha u_{\beta,\beta}^* \\
&-\frac{3(1-\nu)D}{2h^2(1+2\gamma)}(u_\alpha^* + u_{3\alpha}) = f_\alpha^* + R_{\alpha+2}[u_\alpha], \qquad (2.12)\\
&\frac{3(1-\nu)D}{2h^2(1+2\gamma)}(\Delta u_3 + u_{\alpha,\alpha}^*) = f_3^* + R_5[\bar{u}_3].
\end{aligned}$$

Obviously, the equations (2.7) (or (2.8)-((2.11) and type of (2.12) without remainder terms present a full system of KMR type differential equations with respect to functions $u_i(x,y)$ and $u_\alpha^*(x,y)$.

We remark that the non-linear two-dimensional models for Reissner type DEs with layered effects for anisotropic elastic plates first were constructed in [9].

$$w_{tt} + w_{xxxx} - \Big(\alpha + \beta\int\limits_0^l \Big(\frac{\partial w}{\partial x}\Big)^2 dx\Big)\frac{\partial^2 w}{\partial x^2} = f \qquad (2.13)$$

From equation (2.1) if

$$1 + 2\gamma = \frac{3(1-\nu)D}{h^2(2-\nu)\Delta(\mathrm{g}_3^+ - \mathrm{g}_3^-)}\Big(\alpha + \beta\int\limits_0^l \Big(\frac{\partial w}{\partial x}\Big)^2 dx\Big)\frac{\partial^2 w}{\partial x^2},$$

as $\Delta^2 w = \partial_1^4 w$ and suppose $2E^{-1}(1+\nu)\rho\partial_{tt}\Delta w = [w,\varphi] = 0$, immediately follows (2.13).

This example is typical for constructing nonlinear dynamical models of second order accuracy (considering by Ambartsumian, Antmann, Ball, Lamb, Love, Pochhammer, Rayleigh, Timoshenko). See details in [9], ch.1, p. 2.3.

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