NUMERICAL REALIZATION OF BOUNDARY PROBLEMS OF THE THEORY OF ELASTICITY FOR A DISK WITH VOIDS

I. Tsagareli[1], B. Gulua[1,2]

[1]I.Vekua Institute of Applied Mathematics
of Iv. Javakhishvili Tbilisi State University
11 University Str., Tbilisi 0186, Georgia
i.tsagareli@yahoo.com
[2]Sokhumi State University
Politkovskaya str. 61, 0186 Tbilisi, Georgia
bak.gulua@gmail.com

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Abstract

In the present paper, we explicitly solve, in the form of absolutely and uniformly convergent series, a two-dimensional boundary value problem of statics in linear theory elasticity for an isotropic elastic disk consisting of empty pores. The uniqueness theorem for the solution is proved. For a particular problem numerical results are given.

Keywords and phrases: Elastic disk with voids, boundary value problem, explicit solution, numerical results.

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1 Introduction

In recent years it has become topical to investigate the stressed-deformed state of elastic and thermoelastic bodies taking into account their microstructure. One of simple extensions of elasticity theory for processing materials with microstructure is the linear theory of porous materials with voids. In this theory, instead of pores saturated with liquid we consider pores with voids. The volume fraction corresponding to the void volume is assumed to be an independent variable. It is also assumed that voids have no mechanical or energetic significance. Materials containing voids include, for example, rocks and soils, as well as artificially made ceramic and foamy materials.

The fundamental principles of both linear and nonlinear theories of materials with voids were elaborated by Cowin and Nunziato [1,2]. The linear theory of thermoelasticity of elastic materials with empty pores was developed by Iesan [3, 9].

Issues related to this topic are considered, for example, in [3–21]. Fundamental results on the theory of materials with voids and bibliographic information can also be found in the books: [22-26].

From the standpoint of applications, the topical problem is the construction of solutions in an explicit form which is convenient for engineering practice and also enables one to perform numerical analysis of the problems under investigation.

In the present paper we consider static boundary value problem of theory elasticity for the elastic circular disk with voids. Special representations of a general solution of a system of differential equations of the theory of elastic materials with voids are constructed by using harmonic, biharmonic and metaharmonic functions, which make it possible to reduce the initial system of equations to equations of simple structure. This approach facilitates the solution of the original problems. With the help of these representations, the solution of the formulated boundary value problem is obtained explicitly in the form series.

The simple form of the obtained expressions makes it easy to create a program and apply standard programs for the numerical solution of the problem with the help of any of the systems of computer mathematics. The algorithm for the approximate solution of the problem under consideration is based on the calculation of an effective solution at a given point inside the disk. For a particular boundary value problem numerical results are given.

2 Formulation of boundary value problem

Consider an isotropic elastic disk D consisting of empty pores and bounded by a circle S centered at the origin and radius R Assume that the area fraction of void pores at the macropoint $\mathbf{x} = (x_1, x_2)$ is an independent variable. The fraction area of pores changes as a result of body deformation. Denote this change by $\varphi(\mathbf{x})$.

The system of equations of the linear theory of elastic materials in the case of an elastic body with a single distribution of voids has the following form [4]:

$$\mu\,\Delta\mathbf{u}(\mathbf{x}) + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\mathbf{u}(\mathbf{x}) + \beta\,\mathrm{grad}\,\varphi(\mathbf{x}) = 0 \qquad (1)$$

$$(\alpha\,\Delta - \xi)\varphi(\mathbf{x}) - \beta\,\mathrm{div}\mathbf{u}(\mathbf{x}) = 0, \quad \mathbf{x}\in D, \ \ \mathbf{x} = (x_1, x_2), \qquad (2)$$

where $\mathbf{u}(\mathbf{x}) = (u_1, u_2)$ is the displacement vector; λ and μ are Lame constants; α, β and ξ are the constants characterizing the body porosity. Let us formulate our boundary value problems.

Find, in the disk D, a regular vector $U(\mathbf{x}) = (\mathbf{u(x)}, \varphi(\mathbf{x}))$, $(U(\mathbf{x}) \in C^1\bar{D} \cap C^2(D))$, which satisfies the system (1), (2) and of the following conditions on the boundary S:

$$\mathbf{u(z)} = \mathbf{f}(\mathbf{z}), \quad \varphi(\mathbf{z}) = f_3(\mathbf{z}), \quad \mathbf{z} \in S, \tag{3}$$

where $\mathbf{z} = (z_1, z_2) \in S$, $\mathbf{n(z)} = (n_1(\mathbf{z}), n_2(\mathbf{z}))$ is the external normal to S at the point $\mathbf{z}$, $\mathbf{f} = (f_1, f_2), f_3$ are the given functions on S.

$$\boldsymbol{P}(\partial_x, \mathbf{n})\mathbf{U(x)} = \begin{pmatrix} \boldsymbol{P_1}(\partial_x, \mathbf{n})\mathbf{U(x)} \\ \alpha\partial_n\varphi(\mathbf{x}) \end{pmatrix} \tag{4}$$

is the stress vector in the theory of elasticity for porous bodies with voids [4], where

$$\boldsymbol{P_1}(\partial x, \mathbf{n})\mathbf{U(x)} = \mathbf{T}(\partial_x, \mathbf{n})\mathbf{u(x)} + \beta\,\mathbf{n(x)}\,\varphi(\mathbf{x}), \tag{5}$$

and

$$\boldsymbol{T}(\partial_x, \mathbf{n})\mathbf{u(x)} = \mu\,\partial_n\mathbf{u(x)} + \lambda\mathbf{n(x)}\mathrm{div}\,\mathbf{u(x)} + \mu\sum_{i=1}^{2} n_i(\mathbf{x})\mathrm{grad}\,u_i(\mathbf{x})$$

is the stress vector in the classical theory of elasticity, $\partial_n = (\mathbf{n} \cdot \mathrm{grad})$.

3 General representations of solutions of a system of equations

Let us write representations for $\mathbf{u(x)}$ and $\varphi(\mathbf{x})$. We represent a solution of the system of equations (1) and (2) as

$$\begin{aligned} \mathbf{u(x)} &= c_1\overset{\mathbf{0}}{\mathbf{u}}(\mathbf{x}) + c_2\overset{\mathbf{1}}{\mathbf{u}}(\mathbf{x}) \\ \varphi(\mathbf{x}) &= \varphi_1(\mathbf{x}) + \varphi_2(\mathbf{x}). \end{aligned} \tag{6}$$

where φ_1 is a harmonic function: $\Delta\varphi_1 = 0$, and φ_2 is a metaharmonic function with the parameter σ_1^2:

$$\Delta + \sigma_1^2)\varphi_2 = 0; \quad \sigma_1^2 = -\frac{\mu_0\,\xi - \beta^2}{\mu_0\,\alpha}, \;\; \mu_0 = \lambda + 2\mu.$$

Assume

$$\lambda > 0, \;\; \mu > 0, \;\; \alpha > 0, \;\; \mu_0\,\xi - \beta^2 > 0. \tag{7}$$

Taking into account (7), we write:

$$\xi > 0, \;\; \sigma_1^2 < 0, \;\; \sigma_1 = i\sqrt{\frac{\mu_0\,\xi - \beta^2}{\mu_0\alpha}} = i\sigma_0, \;\; i = \sqrt{-1}.$$

c_0 and c_1 are unknown for the time being.

A general solution $\overset{0}{\mathbf{u}} = \left(\overset{0}{\mathrm{u}}_1, \overset{0}{\mathrm{u}}_2\right)$ of the homogeneous equation, corresponding to the nonhomogeneous equation (1) with respect $\mathbf{u(x)}$, is represented as follows

$$\overset{0}{\mathbf{u}}\mathbf{(x)} = \mathrm{grad}\,[\Phi_1(\mathbf{x}) + \Phi_2(\mathbf{x})]\,\mathrm{rot}\Phi_3(\mathbf{x}) + e\tilde{\mathbf{x}}, \tag{8}$$

where the functions Φ_2 and Φ_3 are interrelated by

$$\mu_0\,\mathrm{grad}\,\Delta\Phi_2(\mathbf{x}) + \mu\,\mathrm{rot}\Phi_3(\mathbf{x}) = 0; \tag{9}$$

$\Delta\Phi_1(\mathbf{x}) = 0$, $\Delta\Delta\Phi_2(\mathbf{x}) = 0$, $\Delta\Delta\Phi_3(\mathbf{x}) = 0$, Φ_1, Φ_2, Φ_3 are scalar functions, $\tilde{\mathbf{x}} = (x_2, -x_1)$, $\mathrm{div}\tilde{\mathbf{x}} = 0$, e is the sought coefficient, $\mathrm{rot} = \left(-\dfrac{\partial}{\partial x_2}, \dfrac{\partial}{\partial x_1}\right)$. $\overset{1}{\mathbf{u}} = \left(\overset{1}{\mathrm{u}}_1, \overset{1}{\mathrm{u}}_2\right)$ is one of the particular solutions of the equation (1):

$$\overset{1}{\mathbf{u}}\mathbf{(x)} = -\frac{\beta}{\mu_0}\mathrm{grad}\left(-\frac{1}{\sigma_1^2}\,\varphi_2 + \varphi_0\right), \tag{10}$$

where φ_0 is chosen such that $\Delta\varphi_0 = \varphi_1$. It is obvious that φ_0 is a biharmonic function: $\Delta\Delta\varphi_0 = \Delta\varphi_1 = 0$.

For simplicity, the function φ_1 is chosen such that $\varphi_1 = \mathrm{div}\overset{0}{\mathbf{u}} \equiv \Delta\Phi_2$. Then we can take $\varphi_0 = \Phi_2$.

Let us calculate the values of the coefficients c_0 and c_1 in representation (10). We apply the operator div to the first equality in (10) and compare the obtained expression with divu defined by equation (2). Using (7), we obtain

$$c_0 = -\frac{\mu_0\,\xi - \beta^2}{\mu_0\beta}, \quad c_1 = 1. \tag{11}$$

By an immediate verification we make sure that representations (6) satisfy equations (1) and (2).

4 Uniqueness theorem

For a regular solution $\mathbf{U(x)}$ of equation (1) there holds the Green formula

$$\int\limits_D [E(\mathbf{u},\mathbf{u}) + \beta\,\varphi\,\mathrm{div}\,\mathbf{u}]\,dx = \int\limits_S \mathbf{u}\,[\mathbf{T}(\partial_y, \mathbf{n}) + \beta\,\varphi\mathbf{n}]\,d_yS, \tag{12}$$

where

$$E(\mathbf{u},\mathbf{u}) = (\lambda+\mu)\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\right)^2+\mu\left(\frac{\partial u_1}{\partial x_1}-\frac{\partial u_2}{\partial x_2}\right)^2+\mu\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right)^2,$$

under conditions (7), which is a non negative quadratic form. Multiplying equality (2) by $\varphi(\mathbf{x})$ and integrating over D. After simple transformations, we get

$$\int_D \left[\alpha|grad\varphi|^2+\xi\,\varphi^2+\beta\,\varphi\mathrm{div}\mathbf{u}\right]dx=\int_S \alpha\varphi\frac{\partial\varphi}{\partial n}d_yK. \tag{13}$$

Let $\boldsymbol{U'}=(\boldsymbol{u'},\varphi')$ and $\boldsymbol{U''}=(\boldsymbol{u''},\boldsymbol{\varphi''})$ be two arbitrary solutions of the problem. For their difference $\mathbf{U}=(\mathbf{u},\varphi)=\boldsymbol{U'}-\boldsymbol{U''}$ from (15) and (16), taking into account (7), we get:

$$\varphi(\mathbf{x})=0,\quad \mathbf{E(u,u)}=0.$$

A solution of this equation has the form

$$u_1(\mathbf{x})=-px_2+q_1,\quad u_2(\mathbf{x})=px_1+q_2, \tag{14}$$

where p, q_1, q_2 are arbitrary constants

Since for a homogeneous boundary condition we have $u_1(\mathbf{z})=u_2(\mathbf{z})=0$, $\varphi(\mathbf{z})=0$, then in formulas (14) we must take: $p=q_1=q_2=$ Therefore, for the difference of solutions to the problem, we obtain: $\mathbf{u}(\mathbf{x})=0$, $\mathbf{x}\in D$. Thus, the following assertion are true.

Theorem. *Problem has a unique solution.*

5 Solution of the problem in an explicit form

We write vector $\mathbf{u(x)}$, represented by equalities (6), in terms of the normal and tangent components:

$$\begin{aligned} u_n\mathbf{(x)}&=\frac{\partial}{\partial r}(c_0\Phi_1+c_3\Phi_2+c_4\varphi_2)-\frac{c_0}{r}\frac{\partial}{\partial\vartheta}\Phi_3,\\ u_S\mathbf{(x)}&=\frac{1}{r}\frac{\partial}{\partial\vartheta}(c_0\Phi_1+c_3\Phi_2+c_4\varphi_2)+c_0\frac{\partial}{\partial r}\Phi_3-er, \end{aligned} \tag{15}$$

$$\varphi(\mathbf{x})=\varphi_1+\varphi_2,$$

where $c_3=-\dfrac{\xi}{b}$, $c_4=\dfrac{\beta}{\mu_0\sigma_1^2}$, $\mathbf{x}=(r,\vartheta)$, $r^2=x_1^2+x_2^2$, $\mathbf{x}\in D$.

Using formula (9) and the equality $\varphi_1=\Delta\Phi_2$, the harmonic functions Φ_1 and φ_1, biharmonic functions Φ_2 and Φ_3, and also the metaharmonic function φ_2 contained in (12) are represented in the disc D in the form of series

$$\begin{aligned}
&\varphi_1(\mathbf{x})=\sum_{m=0}^{\infty}\left(\frac{r}{R}\right)^m(\mathbf{X}_{m1}\cdot\boldsymbol{\nu}_m(\vartheta)),\ \varphi_2(\mathbf{x})=\sum_{m=0}^{\infty}\mathbf{I}_m(\sigma_0 r)(\mathbf{X}_{m2}\cdot\boldsymbol{\nu}_m(\vartheta)),\\
&\Phi_2(\mathbf{x})=\frac{R^2}{4}\sum_{m=0}^{\infty}\frac{1}{m+1}\left(\frac{r}{R}\right)^{m+2}(\mathbf{X}_{m1}\cdot\boldsymbol{\nu}_m(\vartheta)),\\
&\Phi_3=\frac{\mu_0 R^2}{4\mu}\sum_{m=0}^{\infty}\frac{1}{m+1}\left(\frac{r}{R}\right)^{m+2}(\mathbf{X}_{m1}\cdot s_m(\vartheta)),\\
&\Phi_1(\mathbf{x})=\sum_{m=0}^{\infty}\left(\frac{r}{R}\right)^m(\mathbf{X}_{m3}\cdot\boldsymbol{\nu}_m(\vartheta)),
\end{aligned}\tag{16}$$

where $\mathbf{X}_{mk}$ is the sought two-component vector, $k=1,2,3$; $\boldsymbol{\nu}_m(\vartheta)=(\cos m\vartheta,\,\sin m\vartheta)$, $s_m(\vartheta)=(-\sin m\vartheta,\,\cos m\vartheta)$; $\mathbf{I}_m(\sigma_0 r)$; is the Bessel function of an imaginary argument, $\mathbf{I}'_m(\sigma_0 r)=\dfrac{\partial}{\partial(\sigma_0 r)}\mathbf{I}_m(\sigma_0 r)$.

We write the boundary conditions (3) in the form of normal and tangential components

$$u_n(\mathbf{z})=f_n(\mathbf{z}),\quad u_s(\mathbf{z})=f_s(\mathbf{z}),\ \ \varphi(\mathbf{z})=f_3(\mathbf{z}),\tag{17}$$

Let the functions f_n, f_s and f_3, f_4, be expanded into Fourier series

$$\begin{aligned}
&f_n(\mathbf{z})=\frac{\boldsymbol{\alpha}_0}{2}+\sum_{m=1}^{\infty}(\boldsymbol{\alpha}_m\cdot\boldsymbol{\nu}_m(\boldsymbol{\psi}))\quad f_s(\mathbf{z})=\frac{\boldsymbol{\beta}_0}{2}+\sum_{m=1}^{\infty}(\boldsymbol{\beta}_m\cdot s_m(\boldsymbol{\psi}))\\
&f_3(\mathbf{z})=\frac{\boldsymbol{\gamma}_0}{2}+\sum_{m=1}^{\infty}(\boldsymbol{\gamma}_m\cdot\boldsymbol{\nu}_m(\boldsymbol{\psi}))
\end{aligned}\tag{18}$$

where $\boldsymbol{\alpha}_m=(\alpha_{m1},\alpha_{m2})$, $\boldsymbol{\beta}_m=(\beta_{m1},\beta_{m2})$, $\boldsymbol{\gamma}_m=(\gamma_{m1},\gamma_{m2})$ are the Fourier coefficients of the functions f_n, f_S, f_3, respectively. We substitute expressions (16) into (15) and pass to the limit as $r\to R$. Using (18), we substitute the results obtained into (17). For $m=0$ we get a linear algebraic system

$$\begin{aligned}
&\frac{c_3}{2}\mathbf{X}_{01}+c_4 s_0\mathbf{I}'_0(\sigma_0 R)\mathbf{X}_{02}+\frac{c_4}{2}\mathbf{X}_{04}=\frac{\alpha_0}{2},\\
&\frac{c_0\mu_0 R^2}{2\mu}\mathbf{X}_{01}-R\mathbf{X}_{03}=\frac{\beta_0}{2},\\
&\mathbf{X}_{01}+\mathbf{I}_0(\sigma_0 R)\mathbf{X}_{02}=\frac{\gamma_0}{2},\quad \mathbf{X}_{03}\equiv e;
\end{aligned}\tag{19}$$

and for each $m = 1, 2, \cdots$ we obtain

$$\begin{aligned}
&\frac{R}{4(m+1)}\left[c_3(m+2)+\frac{c_0\mu_0 m}{\mu}\right]\mathbf{X}_{m1}+c_4\sigma_0\mathbf{I}'_m(\sigma_0 R)\mathbf{X}_{m2}\\
&+\frac{c_0 m}{R}\mathbf{X}_{m3}=\alpha_m,\\
&\frac{R[c_3\mu m+c_0\mu_0 Rm(m+2)]}{4\mu(m+1)}\mathbf{X}_{m1}+\frac{c_4 m}{R}\mathbf{I}_m(\sigma_0 R)\mathbf{X}_{m2}\\
&+\frac{c_0 m}{R}\mathbf{X}_{m3}=\beta_m,\\
&\mathbf{X}_{m1}+\mathbf{I}_m(\sigma_0 R)\mathbf{X}_{m2}=\gamma_m.
\end{aligned} \tag{20}$$

Since the problem has a unique solution, the determinants of systems (19) and (20) are nonzero.

We solve systems (19) and (20) and substitute the obtained values of the vectors $\mathbf{X}_{mk}$ in (16). Using formulas (8), (10) and (6), we obtain the solution of the considered problem.

The conditions $\mathbf{f} \in C^3(K)$, $f_3 \in C^3(S)$ provide the absolute and uniform convergence of the resulting.

6 Numerical solutions

For the numerical solution of the problem, a program was compiled, with the help of which, using formulas (6), (8), (10) and (16), we calculate the values of the components of the vector $\mathbf{U(x)}$. These formulas include infinite series. In practical calculations in series, we leave a m_0 finite number of terms, and remove the infinite part. It is known that if $f(\mathbf{x}) \in C^2(S)$, then the cut-off part (sum) of the Fourier series of the function $f(\mathbf{x})$ is estimated as follows:

$$\sum_{m=m_0+1}^{\infty} f_m < \frac{c'}{m_0^{\frac{3}{2}}}, \quad c' = \text{const},$$

where m_0 is a sufficiently large natural number. In particular, this estimate shows that it is enough to require $m_0 = 100$ and the sum of the cut-off part will be accurate to 10^{-3}.

Standard programs are used to calculate definite integrals (Fourier coefficients) and special functions (Bessel functions).

A cylindrical body was considered, the cross section of which is a disk of radius R, centered at the origin. For the purposes of numerical evaluations, magnesium was chosen as the material, for which the elastic constants and void parameters are as follows (see, for example, Ref. [14]):

$$\lambda = 2.17 \times 10^{10} Nm^{-2},\ \ \mu = 3.278 \times 10^{10} Nm^{-2},\ \ \alpha = 3.688 \times 10^{-5} N,$$

$$\beta = 1.13849 \times 10^{10} Nm^{-2},\ \ \xi = 1.475 \times 10^{10} Nm^{-2}.$$

$$R = 2, \quad r = 1, \quad \vartheta = \vartheta_0 \frac{\pi}{180}, \quad \vartheta_0 = 60.$$

Parameter units are given in SI units. Assuming that the displacements are very small compared to the linear dimensions of the body and, in addition, the change in the empty area fraction is not too large, the functions f_1, f_2, f_3, given on the boundary S of the disk under consideration, were taken in the following form:

$$f_1(\vartheta) = \frac{1}{2}\left(\cos(\vartheta) - \frac{1}{3}\right) \times 10^{-4},\ \ f_2(\vartheta) = R \times (\sin\vartheta + 3) \times 10^{-4},$$

$$f_3(\vartheta) = R \times (\cos\vartheta + 10^{-1}) \times 10^{-2}.$$

The program is implemented with the help of the MathCAD mathematical calculation system. Calculation results:

$$u_1(\mathbf{x}) = 0.053, \quad u_2(\mathbf{x}) = -0.03, \quad \varphi(\mathbf{x}) = 7.373 \times 10^{-3}.$$

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