# THE PROBLEM OF FINDING AN EQUALLY STRONG CONTOUR IN THE CASE OF A VISCOELASTIC SQUARE PLATE 

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#### Abstract

The problem of finding an equally strong contour inside a rectangular viscoelastic plate according to the Kelvin-Voigt model is considered. It is assumed that normal compressive forces with given principal vectors (or constant normal displacements) are applied on the sides of the rectangle by means of a linear absolutely rigid stamp, and the unknown part of the boundary (the desired equal-strength contour) is free from external forces. The equal strength of the desired contour lies in the fact that at each point of the contour the tangential normal stress takes on the same values. To solve the problem, methods of conformal mappings and boundary value problems of analytic functions are used, and the equation of the desired contour, as a function of point and time, is constructed effectively (in an analytical form).

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## 1 Introduction

The problems of finding an equally strong contour both in the plane theory of elasticity and in viscoelasticity can be attributed to an extensive class
of problems of optimizing the form of elastic and viscoelastic bodies [see. 1]). In the case of elasticity theory, the mentioned problems for doubly connected polygonal domains are considered in $[2-4]$.

As is known, the presence of a hole in a plate causes an uneven distribution of stresses near the contour of the holes and the appearance of plastic zones. The tangential normal stress plays an important role in this process. As the hole expands, the values of these stresses increase, and for viscoelastic bodies, arising from the process of stress relaxation, their values decrease over time. Thus, it becomes interesting to regulate the shape and size of the hole so that for each moment of time the stresses mentioned remain constant. This paper is the subject of this work, based on the Kelvin-Voigt model [5].

## 2 Statement of the Problem

Let the middle surface of a viscoelastic isotropic plate on the complex variable plane $z$ occupy a doubly connected domain $S$ bounded by a rectangle and a smooth closed contour $L_{0}$ (see Fig. 1). Let us assume that rectilinear smooth stamps with known main vectors of normal compressive forces act on the sides of the rectangle (or constant normal displacements are given), and the inner part of the boundary (the desired equal-strength contour) is free from external forces. The equal strength of the desired contour lies in the fact that the tangential normal stress acting on it at each point of this contour takes a constant value, i.e. $\sigma_{\vartheta}(\sigma, t)=K_{0}^{*}=$ const (in the general case, the voltages mentioned depend both on the point and on time). The viscoelasticity of the region is understood by the Kelvin-Voigt model.

## 3 Solution of the problemm

Due to the axial symmetry, we restrict ourselves to the consideration of elastic equilibrium on a quarter of the plate only and denote it by $S$ (see Fig. 1).

Let us introduce the notation $L=L_{0} \bigcup L_{1}$, where $L_{0}=A_{5} A_{1} ; \quad L_{1}=$ $\bigcup L_{1}^{(k)}\left(L_{1}^{(k)}=A_{k} A_{k+1}, k=\overline{1,4}\right)$.

Let us present some results of works [6-9], namely, the boundary conditions of the first and second basic problems for a viscoelastic plate according to the Kelvin-Voigt model have the form

$$
\begin{equation*}
\varphi(\sigma, t)+\sigma \overline{\varphi^{\prime}(\sigma, t)}+\overline{\psi(\sigma, t)}=i \int_{A_{1}}^{\sigma}\left(X_{n}+i Y_{n}\right) d s, \quad \sigma \in L_{1} \tag{1}
\end{equation*}
$$



Figure 1:

$$
\begin{gather*}
\begin{array}{c}
\int_{0}^{t}\left[\mathfrak{æ}^{*} e^{k(\tau-t)} \varphi(\sigma, \tau)+e^{m(\tau-t)}\left(\varphi(\sigma, \tau)-\sigma \overline{\varphi^{\prime}(\sigma, \tau)}-\overline{\psi(\sigma, \tau)}\right)\right] d \tau \\
=2 \mu^{*}(u+i v), \quad \sigma \in L_{1} ; \\
\varphi(\sigma, t)+\sigma \overline{\varphi^{\prime}(\sigma, t)}+\overline{\psi(\sigma, t)}=0, \quad \sigma \in L_{0} ; \\
\begin{aligned}
\int_{0}^{t}\left[\mathfrak{æ}^{*} e^{k(\tau-t)} \varphi(\sigma, \tau)+e^{m(\tau-t)}\left(\varphi(\sigma, \tau)-\sigma \overline{\varphi^{\prime}(\sigma, \tau)}-\overline{\psi(\sigma, \tau)}\right)\right] d \tau \\
=0, \quad \sigma \in L_{0} ;
\end{aligned} \\
\operatorname{Re}\left[\varphi^{\prime}(\sigma, t)\right]=\operatorname{Re}[\Phi(\sigma, t)]=\frac{K_{0}^{*}}{4}=K_{0}, \quad \sigma \in L_{0} .
\end{array} \tag{2}
\end{gather*}
$$

where

$$
\begin{equation*}
æ^{*}=\frac{2 \mu^{*}}{\lambda^{*}+\mu^{*}} ; \quad k=\frac{\lambda+\mu}{\lambda^{*}+\mu^{*}} ; \quad m=\frac{\mu}{\mu^{*}}, \tag{6}
\end{equation*}
$$

(here and then the coordinate $t$ is the parameter of the time).
From (1) and (2) we obtain the equality

$$
\begin{equation*}
\Gamma^{*}[\varphi(\sigma, t)]=\mathrm{M}\left[i \int_{A_{1}}^{\sigma}\left(X_{n}+i Y_{n}\right) d s\right]+2 \mu^{*}(u+i v), \quad \sigma \in L_{1} \tag{7}
\end{equation*}
$$

where $\Gamma^{*}$ and M are are operators of the time $t$.

$$
\begin{align*}
& \Gamma^{*}[\varphi(\sigma, t)]=\int_{0}^{t}\left[æ^{*} e^{k(\tau-t)}+2 e^{m(\tau-t)}\right] \varphi(\sigma, \tau) d \tau \\
& \mathrm{M}\left[i \int_{A_{1}}^{\sigma}\left(X_{n}+i Y_{n}\right) d s\right]=\int_{0}^{t} e^{m(\tau-t)}\left[i \int_{A_{1}}^{\sigma}\left(X_{n}+i Y_{n}\right) d s\right] d \tau \tag{8}
\end{align*}
$$

Considering that in the case under consideration

$$
T(\sigma, t)=0, \sigma \in L_{1} ; \quad N(\sigma, t)=T(\sigma, t)=0, \sigma \in L_{0},
$$

$$
v_{n}=v_{n}^{(k)}=\mathrm{const}(k=\overline{1,4}), v_{s}=0, \sigma \in L_{1} ; v_{n}=v_{s}=0, \sigma \in L_{0},
$$

and taking into account the equalities

$$
X_{n}+i Y_{n}=(N+i T) e^{i \alpha(\sigma)}, u+i v=\left(v_{n}+i v_{s}\right) e^{i \alpha(\sigma)}
$$

( $\alpha(\sigma)$ - the angle between the axis $O x$ and the outer normal to the contour $L_{1}$ ), from (7) we obtain

$$
\begin{gather*}
\operatorname{Re}\left[\Gamma^{*}\left[e^{-i \alpha(\sigma)} \varphi(\sigma, t)\right]\right]=C(\sigma) F(t)+2 \mu^{*} v_{n}(\sigma), \quad \sigma \in L_{1} ;  \tag{9}\\
\operatorname{Re}\left[\varphi^{\prime}(\sigma, t)\right]=K_{0}, \quad \sigma \in L_{0}, \tag{10}
\end{gather*}
$$

where

$$
\begin{gathered}
C(\sigma)=\operatorname{Re}\left[i \int_{A_{1}}^{\sigma} N\left(s_{0}\right) e^{i\left[\alpha\left(s_{0}\right)-\alpha(s)\right]} d s_{0}\right]=\sum_{j=1}^{r} N\left(s_{0}\right) \sin \left(\alpha_{j}-\alpha_{r}\right) d s_{0}= \\
=C_{r}=\text { const }, \quad \sigma \in L_{1}^{(r)}, r=\overline{1,4}, \\
F(t)=\frac{1}{m}\left[1-e^{-m t}\right] .
\end{gathered}
$$

Mapping the domain $S$ onto a unit circle using the function $z=\omega^{0}(\zeta)$, and then differentiating (9) along the arc abscissa $s$, taking into account the piecewise constancy (with respect to $\sigma$ ) of the right side of (9), with respect to the function

$$
\Omega(z, t)=\Gamma^{*}\left[\varphi^{\prime}(z, t)-K_{0}\right]=\Gamma^{*}\left[\Phi(z, t)-K_{0}\right],
$$

we obtain the Riemann-Hilbert boundary value problem for the circle

$$
\begin{equation*}
\operatorname{Im} \Omega(\eta, t)=0, \quad \eta \in l_{1} ; \quad \operatorname{Re} \Omega(\eta, t)=0, \quad \eta \in l_{0}, \tag{11}
\end{equation*}
$$

( $l_{1}$ and $l_{0}$ are arcs corresponding to lines $L_{1}$ and $L_{0}$ ).
Problem (11) has only a trivial solution and, thus, we will have

$$
\begin{equation*}
\Gamma^{*}\left[\Phi(z, t)-K_{0}\right]=0 . \tag{12}
\end{equation*}
$$

It is easy to prove that equation (12) has only a trivial solution, and thus for the function we obtain the formula

$$
\begin{equation*}
\varphi(z, t)=K_{0} \cdot z \tag{13}
\end{equation*}
$$

(an arbitrary constant is assumed to be zero).

Taking into account (13), the boundary conditions (1) and (2) are written in the form

$$
\begin{gather*}
2 K_{0} \sigma+\overline{\psi(\sigma, t)}=i \int_{A_{1}}^{\sigma}\left(X_{n}+i Y_{n}\right) d s, \quad \sigma \in L_{1}  \tag{14}\\
\Gamma\left[K_{0} \sigma\right]-\mathrm{M}[\overline{\psi(\sigma, t)}]=2 \mu^{*}(u+i v), \quad \sigma \in L_{1}  \tag{15}\\
2 K_{0} \sigma+\overline{\psi(\sigma, t)}=0, \quad \sigma \in L_{0}  \tag{16}\\
\Gamma\left[K_{0} \sigma\right]-\mathrm{M}[\overline{\psi(\sigma, t)}]=0, \quad \sigma \in L_{0} \tag{17}
\end{gather*}
$$

where $\Gamma$ is a operator os the time.

$$
\begin{equation*}
\Gamma\left[K_{0} \sigma\right]=\int_{0}^{t} æ^{*} K_{0} \sigma e^{k(\tau-t)} d \tau \tag{18}
\end{equation*}
$$

and $M$ is determined by formula (8).
From the boundary conditions (14), (15) and (17), after some transformations, with respect to the function

$$
\begin{equation*}
\Phi_{1}(z, t)=\Gamma\left[K_{0} z\right]+\mathrm{M}[\psi(z, t)] \tag{19}
\end{equation*}
$$

we obtain the boundary conditions

$$
\begin{align*}
& \operatorname{Im} \Phi_{1}(\sigma, t)=0, \quad \sigma \in L_{0} ; \quad \operatorname{Im} \Phi_{1}(\sigma, t)=0, \sigma \in L_{1}^{(1)} \\
& \operatorname{Re} \Phi_{1}(\sigma, t)=\Gamma\left[2 K_{0} a\right]+2 \mu^{*} v_{n}^{(2)}, \sigma \in L_{2}  \tag{20}\\
& \operatorname{Im} \Phi_{1}(\sigma, t)=2 \mu^{*} v_{n}^{(3)}, \quad \sigma \in L_{1}^{(3)} ; \quad \operatorname{Re} \Phi_{1}(\sigma, t)=0, \quad \sigma \in L_{1}^{(4)}
\end{align*}
$$

In addition, for normal displacements $v_{n}^{2}$ and $v_{n}^{3}$ we obtain the formulas

$$
\begin{align*}
& 2 \mu^{*} v_{n}^{(1)}=-\left[\Gamma\left[K_{0} a\right]+\mathrm{M}\left[2 K_{0} a+P / 2\right]\right] \\
& 2 \mu^{*} v_{n}^{(3)}=-\left[\Gamma\left[K_{0} b\right]+\mathrm{M}\left[2 K_{0} b+Q / 2\right]\right] \tag{21}
\end{align*}
$$

Similarly, from (14)-(17), with respect to the function

$$
\begin{equation*}
\Phi_{2}(z, t)=i\left[\Gamma\left[K_{0} z\right]-\mathrm{M}[\psi(z, t)]\right] \tag{22}
\end{equation*}
$$

we have

$$
\begin{align*}
& \operatorname{Im} \Phi_{2}(\sigma, t)=0, \quad \sigma \in L_{0} ; \quad \operatorname{Re} \Phi_{2}(\sigma, t)=0, \sigma \in L_{1} \\
& \operatorname{Im} \Phi_{2}(\sigma, t)=\Gamma\left[K_{0} a\right]+\mathrm{M}\left[2 K_{0} a+P / 2\right] \sigma \in L_{2} ; \\
& \operatorname{Re} \Phi_{2}(\sigma, t)=-\Gamma\left[K_{0} b\right]+\mathrm{M}\left[2 K_{0} b+Q / 2\right] \sigma \in L_{3} ;  \tag{23}\\
& \operatorname{Im} \Phi_{2}(\sigma, t)=0, \quad \sigma \in L_{4} .
\end{align*}
$$

Problems (20) and (23) are problems of the same type.
Let the function $z=\omega(\zeta)$ map conformally the unit semi-circle $D_{0}=$ $\{|\zeta|<1 ; \operatorname{Im} \zeta>0\}$ onto the region $S$. By $a_{k}(k=1, \ldots, 5)$ we denote preimages of the points $A_{k}$ and assume that $a_{1}=1, a_{3}=i, a_{5}=-1$, i.e. the contour $L_{0}$ transforms into the segment $[-1,1]$.

Consider the functions

$$
W_{j}(\zeta, t)=\left\{\begin{array}{l}
\Phi_{j}(\zeta, t), \quad \operatorname{Im} \zeta>0  \tag{24}\\
\Phi_{j *}(\zeta, t), \operatorname{Im} \zeta<0, j=1,2 ;
\end{array}\right.
$$

where $\Phi_{j *}(\zeta, t)=\overline{\Phi_{j}(\bar{\zeta}, t)}$.
On the basis of (20) and (23) we conclude that the function $W_{j}(\zeta, t) \quad(j=$ 1,2 ) is holomorphic in the circle $D=\{|\zeta|<1\}$, continuously extendable up to the boundary $l=\{|\zeta|=1\}$ and satisfies the boundary conditions

$$
\begin{align*}
& W_{1}(\omega, t)-W_{1}\left(\frac{1}{\omega}, t\right)=0, \omega \in l_{1}^{(1)} \\
& W_{1}(\omega, t)+W_{1}\left(\frac{1}{\omega}, t\right)=2 H_{11}, \omega \in l_{1}^{(2)},  \tag{25}\\
& W_{1}(\omega, t)-W_{1}\left(\frac{1}{\omega}, t\right)=2 i H_{12}, \omega \in l_{1}^{(3)} ; \\
& W_{1}(\omega, t)+W_{1}\left(\frac{1}{\omega}, t\right)=0, \omega \in l_{1}^{(4)}, \\
& W_{2}(\omega, t)+W_{2}\left(\frac{1}{\omega}, t\right)=0, \omega \in l_{1}^{(1)} ; \\
& W_{2}(\omega, t)-W_{2}\left(\frac{1}{\omega}, t\right)=2 i H_{21}, \omega \in l_{1}^{(2)}, \\
& W_{2}(\omega, t)+W_{2}\left(\frac{1}{\omega}, t\right)=2 H_{22}, \omega \in l_{1}^{(3)} ;  \tag{26}\\
& W_{2}(\omega, t)-W_{2}\left(\frac{1}{\omega}, t\right)=0, \omega \in l_{1}^{(4)},
\end{align*}
$$

where, taking into account (21), we have

$$
\begin{align*}
& H_{11}=\Gamma\left[K_{0} a\right]-\mathrm{M}\left[2 K_{0} a+P / 2\right] ; \\
& H_{12}=\Gamma\left[K_{0} b\right]+\mathrm{M}\left[2 K_{0} b+Q / 2\right] ;  \tag{27}\\
& H_{21}=\Gamma\left[K_{0} a\right]+\mathrm{M}\left[2 K_{0} a+P / 2\right] ; \\
& H_{22}=-\Gamma\left[K_{0} b\right]+\mathrm{M}\left[2 K_{0} b+Q / 2\right],
\end{align*}
$$

$\left(l_{1}^{(k)}\right.$ are the image of the $\left.L_{1}^{(k)}(k=\overline{1,4})\right)$.
The problems (25) and (26) are of the same type. For the solution of these problems we use the method of conformal sewing (see [10]). Under the sewing function we mean Zhukovski's function $\xi=\zeta+\frac{1}{\zeta}$ which maps
the circle $D$ onto the plane with a cut along the segment $I=[-2 ; 2]$ of the real axis in such a way that the upper semicircle $l^{+}$is mapped onto the upper contour and the lower semicircle $l^{-}$onto the lower contour of the segment $I$. The positive direction on $I$ is assumed to coincide with that of the real axis and consider the inverse function

$$
\zeta(\xi)=\frac{1}{2}\left(\xi-\sqrt{\xi^{2}-4}\right)
$$

where the radical is understood to be its branch, which is positive on the real axis outside the segment $I$. Then we will have

$$
\begin{array}{ll}
\zeta^{+}(\eta)=\frac{1}{2}\left(\eta-\sqrt{\eta^{2}-4}\right), & \omega \in l^{+} \\
\zeta^{-}(\eta)=\frac{1}{2}\left(\eta+\sqrt{\eta^{2}-4}\right), & \frac{1}{\omega} \in l^{-}
\end{array}
$$

Consider the functions

$$
W_{j 0}(\xi, t)=W_{j}[\zeta(\xi), t]=W_{j}\left[\left(\xi-\sqrt{\xi^{2}-4}\right) / 2, t\right], \quad(j=1,2) .
$$

we have

$$
\begin{aligned}
& W_{j}(\omega, t)=W_{j}\left[\frac{1}{2}\left(\eta-\sqrt{\eta^{2}-4}\right), t\right]=W_{j 0}^{+}(\eta, t), \quad \omega \in l^{+}, \\
& W_{j}\left(\frac{1}{\omega}, t\right)=W_{j}\left[\frac{1}{2}\left(\eta+\sqrt{\eta^{2}-4}\right), t\right]=W_{j 0}^{-}(\eta, t), \quad \frac{1}{\omega} \in l^{-}, \quad j=1,2
\end{aligned}
$$

and conditions (25) and (26) are written in the form

$$
\begin{align*}
& W_{10}^{+}(\eta, t)-W_{10}^{-}(\eta, t)=0, \quad \eta \in[\delta ; 2] ; \\
& W_{10}^{+}(\eta, t)+W_{10}^{-}(\eta, t)=2 H_{11}, \quad \eta \in[0 ; \delta] ; \\
& W_{10}^{+}(\eta, t)-W_{10}^{-}(\eta, t)=2 i H_{12}, \quad \eta \in\left[-\delta_{0} ; 0\right] ;  \tag{28}\\
& W_{10}^{+}(\eta, t)+W_{10}^{-}(\eta, t)=0, \quad \eta \in\left[-2 ;-\delta_{0}\right] ; \\
& W_{20}^{+}(\eta, t)+W_{20}^{-}(\eta, t)=0, \quad \eta \in[\delta ; 2] ; \\
& W_{20}^{+}(\eta, t)-W_{20}^{-}(\eta, t)=2 i H_{21}, \quad \eta \in[0 ; \delta], \\
& W_{20}^{+}(\eta, t)+W_{20}^{-}(\eta, t)=2 H_{22}, \quad \eta \in\left[-\delta_{0} ; 0\right] ;  \tag{29}\\
& W_{20}^{+}(\eta, t)-W_{20}^{-}(\eta, t)=0, \quad \eta \in\left[-2 ;-\delta_{0}\right],
\end{align*}
$$

where $-2,-\delta_{0}, 0, \delta, 2$ are the points of the segment $I$ corresponding to the points $a_{k}(k=\overline{1,5})$ under the mapping $\xi=\zeta+\frac{1}{\zeta}$.

We will seek for bounded at infinity solutions of problems (28) and (29) of the class $h\left(-2 ;-\delta_{0} ; 0 ; \delta ; 2\right)[9]$, satisfying the condition

$$
\begin{equation*}
W_{j 0}(\xi, t)=\overline{W_{j 0}(\bar{\xi}, t)} \quad(j=1,2) . \tag{30}
\end{equation*}
$$

The indices of these problems of the mentioned class are equal to $(-2)$.
Necessary and sufficient conditions for the existence of a solution of problems (28) and (29) of class $h\left(-2 ;-\delta_{0} ; 0 ; \delta ; 2\right)$ bounded at infinity, respectively, have the form

$$
\begin{align*}
& i H_{12} \int_{-\delta_{0}}^{0} \frac{d \eta}{\chi_{1}(\eta)}+H_{11} \int_{0}^{\delta} \frac{d \eta}{\chi_{1}(\eta)}=0  \tag{31}\\
& H_{22} \int_{-\delta_{0}}^{0} \frac{d \eta}{\chi_{2}(\eta)}+i H_{21} \int_{0}^{\delta} \frac{d \eta}{\chi_{2}(\eta)}=0 \tag{32}
\end{align*}
$$

where

$$
\begin{align*}
& \chi_{1}(\xi)=\sqrt{(\xi+2)\left(\xi+\delta_{0}\right) \xi(\xi-\delta)} ; \\
& \chi_{2}(\xi)=\sqrt{\left(\xi+\delta_{0}\right) \xi(\xi-\delta)(\xi-2)} . \tag{33}
\end{align*}
$$

Under conditions (31) and (32), the solution of problems (28) and (29) is given by the formulas

$$
\begin{gather*}
W_{10}(\xi, t)=\frac{\chi_{1}(\xi)}{\pi i}\left[i H_{12} \int_{-\delta_{0}}^{0} \frac{d \eta}{\chi_{1}(\eta)(\eta-\xi)}+H_{11} \int_{0}^{\delta} \frac{d \eta}{\chi_{1}(\eta)(\eta-\xi)}\right]  \tag{34}\\
W_{20}(\xi, t)=\frac{\chi_{2}(\xi)}{\pi i}\left[H_{22} \int_{-\delta_{0}}^{0} \frac{d \eta}{\chi_{2}(\eta)(\eta-\xi)}+i H_{21} \int_{0}^{\delta} \frac{d \eta}{\chi_{2}(\eta)(\eta-\xi)}\right] . \tag{35}
\end{gather*}
$$

It is easy to check that the functions $W_{j 0}(\xi, t)=W_{j}[\zeta(\xi), t]=(j=1,2)$ satisfy condition (30).

The integrals involved in formulas (31)-(35) are expressed in terms of elliptic integrals of the first and third kind, namely (see [11])

$$
\begin{gather*}
\int_{-\delta_{0}}^{0} \frac{d \eta}{\chi_{1}(\eta)}=\frac{2}{\sqrt{2\left(\delta+\delta_{0}\right)}} F\left[\frac{\pi}{2} ; \sqrt{\frac{\delta_{0}(\delta+2)}{2\left(\delta+\delta_{0}\right)}}\right] ;  \tag{36}\\
\int_{0}^{\delta} \frac{d \eta}{\chi_{1}(\eta)}=-\frac{2 i}{\sqrt{2\left(\delta+\delta_{0}\right)}} F\left[\frac{\pi}{2} ; \sqrt{\frac{\delta\left(2-\delta_{0}\right)}{2\left(\delta+\delta_{0}\right)}}\right] ; \\
\int_{-\delta_{0}}^{0} \frac{d \eta}{\chi_{1}(\eta)(\eta-\xi)}=-\frac{2}{\xi(\xi-\delta) \sqrt{2\left(\delta+\delta_{0}\right)}} \times \\
\times\left\{-\delta \prod\left[\frac{\pi}{2} ; \frac{\delta_{0}(\xi-\delta)}{\xi\left(\delta+\delta_{0}\right)} ; \sqrt{\frac{\delta_{0}(\delta+2)}{2\left(\delta+\delta_{0}\right)}}\right]+\left(\xi+\delta_{0}\right) F\left[\frac{\pi}{2} ; \sqrt{\frac{\delta_{0}(\delta+2)}{2\left(\delta+\delta_{0}\right)}}\right]\right\} ; \\
\int_{0}^{\delta} \frac{d \eta}{\chi_{1}(\eta)(\eta-\xi)}=\frac{2 i}{\xi\left(\xi+\delta_{0}\right) \sqrt{2\left(\delta+\delta_{0}\right)}} \times  \tag{37}\\
\times\left\{\delta_{0} \prod\left[\frac{\pi}{2} ; \frac{\delta\left(\xi+\delta_{0}\right)}{\xi\left(\delta+\delta_{0}\right)} ; \sqrt{\left.\frac{\delta\left(2-\delta_{0}\right)}{2\left(\delta+\delta_{0}\right)}\right]}\right]+\xi F\left[\frac{\pi}{2} ; \sqrt{\left.\frac{\delta\left(2-\delta_{0}\right)}{2\left(\delta+\delta_{0}\right)}\right]}\right] ;\right.
\end{gather*}
$$

$$
\begin{gather*}
\int_{-\delta_{0}}^{0} \frac{d \eta}{\chi_{2}(\eta)}=\frac{-2 i}{\sqrt{2\left(\delta+\delta_{0}\right)}} F\left[\frac{\pi}{2} ; \sqrt{\frac{(2-\delta) \delta_{0}}{2(\delta) \delta_{0}}}\right] ;  \tag{38}\\
\int_{0}^{\delta} \frac{d \eta}{\chi_{2}(\eta)}=\frac{2}{\sqrt{2\left(\delta+\delta_{0}\right)}} F\left[\frac{\pi}{2} ; \sqrt{\frac{\delta\left(2+\delta_{0}\right)}{2\left(\delta+\delta_{0}\right)}}\right] ; \\
\int_{-\delta_{0}}^{0} \frac{d \eta}{\chi_{2}(\eta)(\eta-\xi)}=\frac{2 i}{\xi(\xi-\delta) \sqrt{2\left(\delta+\delta_{0}\right)}} \times \\
\times\left\{-\delta \Pi\left[\frac{\pi}{2} ; \frac{\delta_{0}(\xi-\delta)}{\xi\left(\delta-\delta_{0}\right)} ; \sqrt{\frac{(2-\delta) \delta_{0}}{2\left(\delta+\delta_{0}\right)}}\right]+\xi F\left[\frac{\pi}{2} ; \sqrt{\left.\frac{(2-\delta) \delta_{0}}{2\left(\delta+\delta_{0}\right)}\right]}\right] ;\right.  \tag{39}\\
\int_{0}^{\delta} \frac{d \eta}{\chi_{2}(\eta)(\eta-\xi)}=\frac{-2}{\xi\left(\xi+\delta_{0}\right) \sqrt{2\left(\delta+\delta_{0}\right)}} \times \\
\times\left\{\delta_{0} \Pi\left[\frac{\pi}{2} ; \frac{\delta\left(\xi+\delta_{0}\right)}{\xi\left(\delta+\delta_{0}\right)} ; \sqrt{\frac{\delta\left(2+\delta_{0}\right)}{2\left(\delta+\delta_{0}\right)}}\right]+(\xi-\delta) F\left[\frac{\pi}{2} ; \sqrt{\frac{\delta\left(2+\delta_{0}\right)+}{2\left(\delta+\delta_{0}\right)}}\right]\right\} ;
\end{gather*}
$$

where

$$
\begin{gathered}
F[\varphi ; k]=\int_{0}^{\phi} \frac{d \varphi}{\sqrt{1-k^{2} \sin ^{2} \varphi}} ; \\
\prod(\varphi, n, k)=\int_{0}^{\varphi} \frac{d \varphi}{\left(1-n \sin ^{2} \varphi\right) \sqrt{1-k^{2} \sin ^{2} \varphi}}
\end{gathered}
$$

are elliptic integrals of the first and third kind, respectively.
If we are satisfied with approximations

$$
F\left[\frac{\pi}{2} ; k\right] \approx \frac{\pi}{2}\left(1+\frac{k^{2}}{4}\right) ; \quad \prod\left[\frac{\pi}{2} ; n ; k\right] \approx \frac{\pi}{2}\left(1+\frac{k^{2}}{4}+\frac{n}{2}\right)
$$

conditions (31) and (32) are written in the form

$$
\begin{aligned}
& H_{12}\left[1+\frac{\delta_{0}(\delta+2)}{8\left(\delta+\delta_{0}\right)}\right]-H_{11}\left[1+\frac{\delta\left(2-\delta_{0}\right)}{8\left(\delta+\delta_{0}\right)}\right]=0 ; \\
& H_{22}\left[1+\frac{\delta_{0}(2-\delta)}{8\left(\delta+\delta_{0}\right)}\right]-H_{21}\left[1+\frac{\delta\left(2+\delta_{0}\right)}{8\left(\delta+\delta_{0}\right)}\right]=0 .
\end{aligned}
$$

Under these conditions, the functions $W_{10}(\xi, t)$ and $W_{20}(\xi, t)$ are given by the formulas

$$
\begin{aligned}
& W_{10}(\xi, t)=\frac{\delta_{0} \delta_{1}(\xi)}{\left[2\left(\delta+\delta_{0}\right)\right]^{3 / 2} \xi^{2}}\left(H_{12}+H_{11}\right) ; \\
& W_{20}(\xi, t)=-\frac{\delta_{0} \delta \chi_{2}(\xi)}{\left[2\left(\delta+\delta_{0}\right)\right]^{3 / 2} \xi^{2}}\left(H_{22}+H_{21}\right) .
\end{aligned}
$$

After finding the functions $W_{j 0}(\xi, t)(j=1,2)$, to determine the conformally mapping function $z=\omega[\zeta(\xi), t]=\omega_{0}(\xi, t)$, based on (18), (19) and (22) we obtain the integral equation

$$
\begin{equation*}
\mathfrak{æ}^{*} \int_{0}^{t} e^{k \tau} \omega_{0}(\xi, \tau) d \tau=e^{k t} N(\xi, t), \tag{40}
\end{equation*}
$$

where

$$
\begin{equation*}
N(\xi, t)=\frac{1}{2 K_{0}}\left[W_{10}(\xi, t)-i W_{20}(\xi, t)\right] \tag{41}
\end{equation*}
$$

From (40) by differentiation with respect to $t$ we obtain

$$
\omega_{0}(\xi, t)=\frac{1}{æ^{*}}[k N(\xi, t)+\dot{N}(\xi, t)],
$$

where $N(\xi, t)$ is defined by formula (41) and $\dot{N}(\xi, t)$ denotes differentiation with respect to $t$.

## References

1. Banichuk N.V. Optimization of forms of elastic bodies (Russian). Nauka, Moscow, 1980.
2. Bantsuri R. One mixed problem of the plane theory with a partially unknown boundary. Proc. A. Razmadze Math. Inst. 140 (2006), 9-16.
3. Kapanadze G. On one problem of the plane theory of elasticity with a partially unknown boundary. Proc. A. Razmadze Math. Inst. 143 (2007), 61-71.
4. Bantsuri R., Kapanadze G. The problem of finding a full-strength contour inside the polygon. Proc. A. Razmadze Math. Inst. 163 (2013), 1-7.
5. Rabotnov Yu.N. Elements of continuum mechanics of materials with memory (Russian). Nauka, Moscow, 1977.
6. Shavlakadze N., Kapanadze G., Gogolauri L. About one contact problem for a viscoelastic halfplate. Transactions of A. Razmadze Math. Inst., 173, 1 (2019), 103-110.
7. Kapanadze G., Gogolauri L. About one contact problem for a viscoelastic halfplate. Transactions of A. Razmadze Math. Inst., 174, 3 (2020), 405-411.
8. Muskhelishvili N. I. Some Basic Problems of the Mathematical Theory of Elasticity. Fundamental equations, plane theory of elasticity, torsion and bending. Fifth revised and enlarged edition (Russian). Nauka, Moscow, 1966.
9. Muskhelishvili N. I. Singular Integral Equations (Russian). Boundary value problems in the theory of function and some applications of them to mathematical physics. Third, corrected and augmented edition. Nauka, Moscow, 1968.
10. Lavrent'ev M. A., Shabat B. V. Methods of the theory of functions of a complex variable (Russian). Nauka, Moscow, 1973.
11. Prudnikov A.P., Brychkov Ju.A., Marichev O.I. Integrals and series. Elementary functions. Nauka, Moscow, 1981.
