ON ONE METHOD FOR SOLVING OF A STATIC BEAM PROBLEM

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Abstract

In the article, the variational-iterative method is used to solve a boundary value problem that describes the static state of a beam. The error of the method is estimated and its effectiveness is checked by an example.

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1 Introduction

Let us consider the equation

$$u''''(x) - a\left(\int_0^L u'^2(x)dx\right)u''(x) = f(x), \quad 0 < x < L, \tag{1}$$

with the boundary conditions

$$u(0) = u(L) = 0, \quad u''(0) = u''(L) = 0. \tag{2}$$

Here $a(\lambda) \geq \text{const} > 0$, $0 \leq \lambda < \infty$, and $f(x)$, $0 < x < L$, are the given functions. $u(x)$, $0 \leq x \leq L$, is the unknown function and L is some constant.

Equation (1) is a particular case of the equation

$$u_{tt}+u_{xxxx}-m\biggl(\int_0^L u_x^2dx\biggr)u_{xx}=f(x,t), \qquad m(\lambda)\geq \text{const}>0, \tag{3}$$

which for $m(\lambda)=\alpha+\beta\lambda$ and $f(x,t)=0$, was proposed by Woinowsky-Krieger [13] in 1950. Equation (3) describes the deflection of an extensible dynamic beam with hinged ends. In 1876, D'Alembert's classical model was generalized by Kirchhoff [6], which introduced an additional nonlinear term (in equation (3) the factor of u_{xx}). Therefore, equations (3) and (1) are often called Kirchhoff-type equations.

Approximate methods for solving of Kirchhoff type static beam equations are used in the works of quite a number of researchers. For example, we may refer to the published works of Dang and Luan [2], Ohm et al. [8], Peradze [9], Ren and Tian [11], Zhuang and Ren [14], and the papers of Berikelashvili et al. [1], Dang and Nguyen [3, 4], Ma [7], Peradze [10], Tsai [12] and others.

In presented work we construct a new numerical algorithm and estimate its error for problem (1), (2). Applying the Galerkin method we reduce the problem (1), (2) to the system of nonlinear equations, which we solve by using the Newton iterative method. A numerical experiment is given to illustrate the effectiveness of the proposed algorithm.

2 The algorithm

Let

$$f(x)\in L^2(0,L) \quad \text{and} \quad |f_i|\leq\frac{\tau}{i^m}\,, \quad i=1,2,\dots\,, \tag{4}$$

where

$$f_i=\frac{2}{L}\int_0^L f(x)\sin\frac{i\pi x}{L}\,dx,$$

while τ and m are some known positive constants. The solution can be written in the form

$$u(x)=\sum_{i=1}^{\infty}u_i\sin\frac{i\pi x}{L}\,, \tag{5}$$

where the coefficients u_i will satisfy the system

$$\biggl[\Bigl(\frac{i\pi}{L}\Bigr)^4+\Bigl(\frac{i\pi}{L}\Bigr)^2 a\biggl(\frac{L}{2}\sum_{j=1}^{\infty}\Bigl(\frac{j\pi}{L}\Bigr)^2u_j^2\biggr)\biggr]u_i=f_i, \quad i=1,2,\dots\,. \tag{6}$$

We will seek an approximate solution in the form of the following finite series

$$u_n(x) = \sum_{i=1}^{n} u_{ni} \sin \frac{i\pi x}{L}\,, \tag{7}$$

and the coefficients u_{ni} we will find from the system

$$\left[\left(\frac{i\pi}{L}\right)^4 + \left(\frac{i\pi}{L}\right)^2 a\left(\frac{L}{2}\sum_{j=1}^{n}\left(\frac{j\pi}{L}\right)^2 u_{nj}^2\right)\right] u_{ni} = f_i, \quad i = 1, 2, \dots, n. \tag{8}$$

Let us write the system (8) in the vector form

$$\boldsymbol{\varphi}(\mathbf{u}_n) = \mathbf{0}, \tag{9}$$

where

$$\mathbf{u}_n = (u_{ni})_{i=1}^{n}, \quad \boldsymbol{\varphi}(\mathbf{u}_n) = (\varphi_i(u_{n1}, u_{n2}, \dots, u_{nn}))_{i=1}^{n},$$

$$\varphi_i(u_{n1}, u_{n2}, \dots, u_{nn}) = \left[\left(\frac{i\pi}{L}\right)^4 + \left(\frac{i\pi}{L}\right)^2 a\left(\frac{L}{2}\sum_{j=1}^{n}\left(\frac{j\pi}{L}\right)^2 u_{nj}^2\right)\right] u_{ni} - f_i,$$
$$i = 1, 2, \dots, n. \tag{10}$$

To the system (9) we apply the Newton iterative method. Let us denote the k-th iteration approximation of u_{ni} by $u_{ni,k}$. Then we can write

$$\mathbf{u}_{n,k+1} = \mathbf{u}_{n,k} - J^{-1}(\mathbf{u}_{n,k})\,\boldsymbol{\varphi}(\mathbf{u}_{n,k}), \quad k = 0, 1, \dots, \tag{11}$$

where we use the following notation for the vectors $\mathbf{u}_{n,k}$, $\boldsymbol{\varphi}(\mathbf{u}_{n,k})$ and the Jacobi matrix $J(\mathbf{u}_{n,k})$

$$\begin{aligned} &\mathbf{u}_{n,k} = (u_{ni,k})_{i=1}^{n}, \quad \boldsymbol{\varphi}(\mathbf{u}_{n,k}) = (\varphi_{ni,k})_{i=1}^{n}, \\ &J(\mathbf{u}_{n,k}) = diag(d_{1,k}, d_{2,k}, \dots, d_{n,k}) + \boldsymbol{v}\boldsymbol{v}^T, \end{aligned} \tag{12}$$

while

$$\varphi_{ni,k} = d_{i,k} u_{ni,k} - f_i, \quad d_{i,k} = \left(\frac{i\pi}{L}\right)^4 + \alpha_k \left(\frac{i\pi}{L}\right)^2, i = 1, 2, \dots, n,$$
$$\boldsymbol{v} = (\alpha_k' L)^{\frac{1}{2}} \left(\left(\frac{i\pi}{L}\right)^2 u_{ni,k}\right)_{i=1}^{n},$$
$$\alpha_k = a\left(\frac{L}{2}\sum_{i=1}^{n}\left(\frac{i\pi}{L}\right)^2 u_{ni,k}^2\right), \quad \alpha_k' = a'\left(\frac{L}{2}\sum_{i=1}^{n}\left(\frac{i\pi}{L}\right)^2 u_{ni,k}^2\right).$$

3 The algorithm error

For a function $w(x) \in \overset{\circ}{W}{}_2^1(0, L)$ the norm is defined as

$$\|w(x)\|_{\overset{\circ}{W}{}_2^1(0,L)} = \left(\int_0^L w'^2(x)dx \right)^{\frac{1}{2}}.$$

The algorithm error we define as the difference

$$u(x) - u_{n,k}(x). \tag{13}$$

Suppose, $\|w\| = \max\limits_{1\le i\le n} |w_i|$ and $\|T\| = \max\limits_{1\le i\le n} \sum\limits_{j=1}^{n} |t_{ij}|$, where $w = (w_i)_{i=1}^n$ and $T = (t_{ij})_{i,j=1}^n$ denote the vector and matrix norms, respectively. Further, let us introduce the parameters $\Delta_{ij} = \delta_{ij} - \gamma_0 \frac{1}{d_j} \left(\frac{\pi}{L}\right)^2 u_{ni,0}u_{nj,0}$, $i, j = 1, 2, \dots, n$, where δ_{ij} is the Kronecker symbol, $\gamma_0 = \alpha'_0 \Bigg(\frac{1}{L} \cdot \left(\frac{L}{\pi}\right)^4 + \alpha'_0 \sum_{i=1}^n (i^2 u_{ni,0})^2 \frac{1}{d_{i,0}} \Bigg)^{-1}$ and $\mathbf{u}_{n,0} = (u_{ni,0})_{i=1}^n$ is the initial approximation vector. Applying Kantorovich's result [5] for the Newton iterative method to the (11) we obtain following result.

Lemma 1. $\boldsymbol{\varphi}(\mathbf{z}) \in C^2(R^n)$ *and if*

$$d_h(\mathbf{u}_{n,0}) = \left\{ \mathbf{u} = (u_i)_{i=1}^n \mid \max_{1\le i\le n} |u_i - u_{ni,0}| \le h \right\} \subset R^n$$

then

a. There exists the matrix $J^{-1}(\mathbf{u}_{n,0})$ *and the equality* $\|J^{-1}(\mathbf{u}_{n,0})\| = p_0$ *holds, where*

$$p_0 = \max_{1\le i\le n} \frac{1}{d_{i,0}} \sum_{j=1}^{n} |\Delta_{ij}|; \tag{14}$$

b. by (12) *the equality* $\|J^{-1}(\mathbf{u}_{n,0})\boldsymbol{\varphi}(\mathbf{u}_{n,0})\| = q_0$ *is valid, where*

$$q_0 = \max_{1\le i\le n} \left| \sum_{j=1}^{n} \Delta_{ij} \left(u_{nj,0} - \frac{f_j}{d_{j,0}} \right) \right|; \tag{15}$$

c.

$$\sum_{l=1}^{n} \left| \frac{\partial^2 \varphi_i(\mathbf{u})}{\partial u_j \, \partial u_l} \right| \le r, \quad i, j = 1, 2, \dots, n, \quad \mathbf{u} \in d_h(\mathbf{u}_{n,0}), \ 2q_0 \le h.$$

Using Lemma 1, for the algorithm error estimate we get following result

Theorem 1. *Suppose* $a(\lambda) \in C^2[0,\infty)$, $a(\lambda) \geq \alpha > 0$, $a'(\lambda) \geq 0$, $0 \leq \lambda < \infty$ *and the conditions* (4) *are fulfilled. Also, Let* $n > 1$, $s_0 = 2np_0q_0r \leq 1$ *and*

$$c_1 = \frac{1}{2m+3}, \qquad c_2 = \frac{1}{\alpha + \left(\frac{\pi}{L}\right)^2} M_1, \qquad c_3 = \pi q_0 \Big(\frac{1}{12L}\Big)^{\frac{1}{2}},$$

where

$$M_1 = \max_{0 \leq \lambda \leq \rho_{1,\infty}} (a'(\lambda)).$$

Then the algorithm error is estimated as follows

$$\begin{aligned}
\left\|u(x) - u_{n,k}(x)\right\|_{\overset{\circ}{W}{}_2^1(0,L)} &\leq \left\|\Delta u_n(x)\right\|_{\overset{\circ}{W}{}_2^1(0,L)} + \left\|\Delta u_{n,k}(x)\right\|_{\overset{\circ}{W}{}_2^1(0,L)} \\
&\leq \left(a_1^{-1}\left(\frac{1}{(n+1)^{2m+3}}\left(c_1 + \frac{1}{n+1}\right)\right)\right)^{\frac{1}{2}} \\
&\quad + c_2\left(a_1^{-1}\left(1 + c_1\left(1 - \frac{1}{n^{2m+3}}\right)\right)\right)^{\frac{1}{2}} \\
\times a_1^{-1}\left(\frac{1}{(n+1)^{2m+3}}\left(c_1 + \frac{1}{n+1}\right)\right) &+ c_3\left(2n^3 + 3n^2 + n\right)^{\frac{1}{2}} s_0^{2^k-1}\Big(\frac{1}{2}\Big)^{k-1},
\end{aligned}$$

where

$$a_1(\lambda) = \frac{8}{\tau^2 L}\left(\frac{\pi}{L}\right)^4 \lambda\, a(\lambda).$$

4 Numerical realization

Let $a(\lambda) = \exp(\lambda)$ and consider the following problem

$$u''''(x) - \exp\left(\int_0^1 u'^2(x)dx\right) u''(x) = 700\left(\sin(\pi x^2) - \sin(\pi x)\right),$$

$$0 < x < 1, \qquad u(0) = u(1) = 0, \quad u''(0) = u''(1) = 0.$$

The exact solution is unknown. We solve the problem for four values of parameter n, $n = 5, 10, 20, 40$. The number of iterations was 16 in each case.

Since the exact solution of the problem is unknown, we will use the another method to estimate the error (see [11]). Consider the general formulation of problem (1), (2). On the left-hand side of equation (6), we replace $u_i, i = 1, 2, ..., n$, by $u_{ni,k}$, while in the sum ∞ we replace by n. We also move f_i from right to left. As a result, on the left-hand side of the original expression, we get a value that in the general case will be different from zero and can serve as a characteristic of the degree of closeness of the approximate solution to the exact one. We call the function

$$r_{n,k} = \sum_{i=1}^{n} \left\{ \left[\left(\frac{i\pi}{L} \right)^4 + \left(\frac{i\pi}{L} \right)^2 a \left(\frac{L}{2} \sum_{j=1}^{n} \left(\frac{j\pi}{L} \right)^2 u_{nj,k}^2 \right) \right] u_{ni,k} - f_i \right\} \sin \frac{i\pi x}{L}$$

the algorithm error. We will use this algorithm error definition in the considered example. The $L^2(0,1)$-norm of the algorithm error is denoted by $\theta_{n,k}$. So, $\theta_{n,k} = ||r_{n,k}||_{L^2(0,1)}$. The values of $\theta_{n,k}$ for $n = 5, 10, 20, 40$ and $k = 0, 10, 16$ are given in Table 1.

Table 1: Norm of the algorithm error

k	0	10	16
$\theta_{5,k}$	230.17391094294	196.391933108218	2.89779556697324
$\theta_{10,k}$	230.17391094294	196.370932199150	0.33440407124641
$\theta_{20,k}$	230.17391094294	196.370657850696	0.06367143922639
$\theta_{40,k}$	230.17391094294	196.370647884763	0.01183014996018

Note that if $k \geq 16$, then $|\theta_{n,k} - \theta_{n,k+1}| < 10^{-14}$. Thus, in this example, a further decrease of the error should occur by increasing the parameter n.

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